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PSEUDOSPECTRAL METHODS FOR THE STABILITY OF PERIODIC SOLUTIONS OF DELAY MODELS

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DAVIDE LIESSI

SUPERVISORE
DIMITRI BREDÀ

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Davide Liessi

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COLOPHON

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CONTACTS

✉ liessi.davide@spes.uniud.it

✉ <http://users.dimi.uniud.it/~davide.liessi/>

CODES

✉ <https://git.dimi.uniud.it/davide.liessi/eigTMNc>

✉ <https://github.com/dliessi/eigTMNc>

✉ <https://gitlab.com/dliessi/eigTMNc>

ABSTRACT

This thesis is devoted to studying numerical tools for investigating the stability properties of periodic solutions of certain classes of delay equations, namely retarded functional differential equations (RFDEs), renewal equations (REs) and systems of coupled REs and RFDEs.

Theoretical and numerical tools for RFDEs are available in the literature. On the other hand, for REs, and a fortiori for coupled equations, the theoretical landscape is incomplete and there are so far no numerical methods for the stability of periodic solutions. This lack of results, along with the great importance of delay equations in applications ranging from engineering to mathematical biology, motivates the research line pursued during my PhD studies.

The approach to numerically determining the stability properties of periodic solutions is based on the link provided by the principle of linearized stability between the local stability properties of a chosen periodic solution and the stability properties of the equation linearized around said solution. These are in turn connected by Floquet theory to the positions in the complex plane of the eigenvalues of the monodromy operator, i.e., the operator that shifts solutions of the linearized equation by one period in time.

The theory for RFDEs is an application of a more general abstract framework for delay equations based on sun-star calculus; both are presented in [DIEKMANN, VAN GILS, VERDUYN LUNEL, AND WALTHER, *Delay Equations*, Springer, 1995]. The sun-star calculus was applied to REs and coupled equations in [DIEKMANN, GETTO, AND GYLLENBERG, *SIAM J. Math. Anal.* 39 (2008)] with the purpose of studying the stability properties of equilibria. In light of the results contained therein, a preliminary study of the extension of Floquet theory and the principle of linearized stability to REs is presented here, along with a summary of the abstract theory and its application to RFDEs, providing substantial insights on the attainability of the theory for REs.

The main original contribution of this thesis is the development of a numerical method, based on the ideas for RFDEs of [BREDÁ, MASET, AND VERMIGLIO, *SIAM J. Numer. Anal.* 50 (2012)], for approximating the eigenvalues of the monodromy operator of linear periodic REs and coupled equations via pseudospectral discretization.

Pseudospectral techniques are based on finite-dimensional approximation by algebraic, possibly orthogonal, polynomial collocation. Their main advantage is that, given sufficient smoothness of the involved functions, pseudospectral methods exhibit an infinite convergence order, hence allowing to attain a satisfactory accuracy with relatively low dimensions of the discretizations and, consequently, low computational costs. This is especially important since it allows for accurate and affordable robust analyses of stability and bifurcations. For the methods presented in this thesis, it is proved

that under suitable conditions on the regularity of the solutions the convergence has infinite order. For RFDEs it can be easily proved that those conditions are fulfilled. For REs and coupled equations it is not clear whether they are attainable, but the numerical tests suggest that in practice the infinite order of convergence is usually achieved.

Extending the method from RFDEs to REs and coupled equations was far from trivial, even though the structure of the proofs turned out to be quite similar. Indeed, due to the delicate interplay between the choices of function spaces and the properties of regularizing the solutions attainable by the equations, several proof lines were attempted, rendering the proof rather involved before being able to simplify it again. A valuable outcome of these various attempts is the uniform structure of the proofs for the three types of equations, together with a better understanding of the interaction between the different aspects of the proofs. As a further result of this process, the convergence proof for RFDEs is restated in this thesis resorting to absolutely continuous rather than Lipschitz continuous functions, which to the best of my knowledge is the least degree of regularity required to complete the proof.

As in the case of RFDEs, the method is general enough that it can be applied to compute the eigenvalues of any evolution operator and not only of monodromy operators. Thus, it can also be applied to study the stability of equilibria by computing the eigenvalues of any evolution operator of the equation linearized around the chosen equilibrium. Moreover, the provided discretization technique may be useful beyond the approximation of eigenvalues, e.g., for computing Lyapunov exponents for generic (nonautonomous) linear systems, in the same spirit of [BREDÁ AND VAN VLECK, Numer. Math. 126 (2014)] for RFDEs. Although this case is not covered by the convergence proofs provided in this thesis, this approach was applied in [BREDÁ, DIEKMANN, LIESSI, AND SCARABEL, Electron. J. Qual. Theory Differ. Equ. 65 (2016)] with promising results.

The thesis contains some numerical tests on equilibria and periodic solutions of nonlinear REs and coupled equations, which validate the theoretical results, demonstrate the effectiveness of the method and provide some revealing hints towards the attainability of the infinite order of convergence and the soundness of Floquet theory and of the principle of linearized stability. Furthermore, an implementation of the method compatible with both MATLAB and Octave is provided.

To summarize, the main contribution of this thesis is that it is now possible to perform stability and bifurcation analyses of periodic orbits of REs and coupled REs/RFDEs by means of the new numerical tool provided here. This constitutes an important step forward in the long way to being able to study realistic models such as the consumer–resource model for the *Daphnia magna* water flea feeding on algae [DIEKMANN, GYLLENBERG, METZ, NAKAOKA, AND DE ROOS, J. Math. Biol. 61 (2010)].

SOMMARIO

Questa tesi è dedicata allo studio di strumenti numerici per investigare le proprietà di stabilità di soluzioni periodiche di alcune classi di equazioni con ritardo, ovvero le equazioni funzionali differenziali con ritardo (RFDE), le equazioni di rinnovo (RE) e sistemi di RE e RFDE accoppiate.

In letteratura si possono trovare diversi strumenti teorici e numerici per RFDE. Per le RE, invece, e a maggior ragione per le equazioni accoppiate, il panorama teorico è incompleto e non sono disponibili finora metodi numerici per la stabilità delle soluzioni periodiche. La linea di ricerca del mio percorso di dottorato è motivata da queste mancanze, oltre che dalla grande importanza delle equazioni con ritardo nelle applicazioni, dall'ingegneria alla biomatematica.

L'approccio proposto per determinare numericamente le proprietà di stabilità delle soluzioni periodiche è basato sul principio di stabilità linearizzata, che lega la stabilità locale di una data soluzione periodica e le proprietà di stabilità dell'equazione linearizzata attorno a tale soluzione. Grazie alla teoria di Floquet, queste sono a loro volta collegate alla posizione nel piano complesso degli autovalori dell'operatore di monodromia, ovvero l'operatore che trasla di un periodo nel tempo le soluzioni dell'equazione linearizzata.

La teoria per le RFDE è l'applicazione di una teoria astratta più generale per le equazioni con ritardo basata sul calcolo sun-star; entrambe sono presentate in [DIEKMANN, VAN GILS, VERDUYN LUNEL E WALTHER, *Delay Equations*, Springer, 1995]. Il calcolo sun-star è stato applicato alle RE e alle equazioni accoppiate in [DIEKMANN, GETTO E GYLLENBERG, *SIAM J. Math. Anal.* 39 (2008)] allo scopo di studiare le proprietà di stabilità degli equilibri. Alla luce di tali risultati, questa tesi presenta uno studio preliminare dell'estensione della teoria di Floquet e del principio di stabilità linearizzata alle RE, insieme a un compendio della teoria astratta e della sua applicazione alle RFDE, fornendo degli indizi importanti sulla verificabilità della teoria per le RE.

Il contributo originale più importante della tesi è lo sviluppo di un metodo numerico, basato sulle idee per le RFDE di [BREDÁ, MASET E VERMIGLIO, *SIAM J. Numer. Anal.* 50 (2012)], per approssimare gli autovalori dell'operatore di monodromia per RE e equazioni accoppiate lineari periodiche attraverso una discretizzazione pseudospettrale.

Le tecniche pseudospettrali consistono nell'approssimazione a dimensione finita attraverso una collocazione polinomiale basata su polinomi algebrici, eventualmente ortogonali. Il vantaggio principale è dato dal fatto che, se le funzioni coinvolte sono sufficientemente lisce, i metodi pseudospettrali convergono con ordine infinito, fornendo così risultati accurati con dimensioni della discretizzazione relativamente basse e, di conseguenza, basso costo computazionale. Ciò è particolarmente importante, dato che permette di effettuare analisi robuste di stabilità e biforcazioni in modo accurato e senza

un costo computazionale proibitivo. Per i metodi presentati in questa tesi, si dimostra che a certe condizioni sulla regolarità delle soluzioni la convergenza avviene con ordine infinito. Per le RFDE si dimostra facilmente che tali condizioni valgono. Per le RE e le equazioni accoppiate non è chiaro se siano verificabili, ma i test numerici suggeriscono che nella pratica solitamente l'ordine di convergenza infinito possa essere ottenuto.

Estendere il metodo dalle RFDE alle RE e alle equazioni accoppiate non è banale, sebbene la struttura delle dimostrazioni si sia rivelata abbastanza simile. Infatti, a causa delle delicate relazioni tra le scelte degli spazi di funzioni e le proprietà di regolarizzazione delle soluzioni verificabili dalle equazioni, è stato necessario tentare diverse linee dimostrative, rendendo la dimostrazione piuttosto complicata, prima che fosse possibile semplificarla nuovamente. Un importante risultato di questi diversi tentativi è la struttura uniforme delle dimostrazioni per i tre tipi di equazioni, nonché una più profonda comprensione delle interazioni tra i diversi aspetti delle dimostrazioni. Un ulteriore risultato di questo processo è la riproposizione in questa tesi della dimostrazione di convergenza per le RFDE impiegando funzioni assolutamente continue, invece che lipschitziane, il che, a quanto mi risulta, è il minimo grado di regolarità che è necessario richiedere per poter completare la dimostrazione.

Come nel caso delle RFDE, il metodo non è limitato agli operatori di monodromia, ma si può applicare al calcolo degli autovalori di qualunque operatore di evoluzione. Perciò, è possibile applicarlo allo studio della stabilità di equilibri calcolando gli autovalori di un qualunque operatore di evoluzione dell'equazione linearizzata attorno all'equilibrio scelto. Inoltre, questa tecnica di discretizzazione può risultare utile al di là dell'approssimazione degli autovalori, per esempio per calcolare gli esponenti di Lyapunov per sistemi lineari generici (non autonomi), nella stessa ottica di [BRED A E VAN VLECK, Numer. Math. 126 (2014)] per le RFDE. Nonostante questo caso non rientri nelle dimostrazioni di convergenza presentate in questa tesi, questo approccio è stato applicato in [BRED A, DIEKMANN, LIESSI E SCARABEL, Electron. J. Qual. Theory Differ. Equ. 65 (2016)] fornendo risultati promettenti.

La tesi contiene alcuni test numerici su equilibri e soluzioni periodiche di RE e equazioni accoppiate non lineari, che forniscono una validazione sperimentale dei risultati teorici, mostrano l'efficacia del metodo e suggeriscono che effettivamente l'ordine di convergenza infinito si possa ottenere e la teoria di Floquet e il principio di stabilità linearizzata siano validi. Inoltre, è disponibile un'implementazione del metodo compatibile sia con MATLAB sia con Octave.

Riassumendo, il contributo principale della tesi è un nuovo strumento numerico per l'analisi di stabilità e biforcazioni di orbite periodiche di RE e RE/RFDE accoppiate, che in precedenza non era possibile. Ciò costituisce un importante passo avanti sulla lunga strada verso la possibilità di studiare modelli realistici, come, per esempio, il modello consumatore-risorsa per la pulce d'acqua *Daphnia magna* che si nutre di alghe [DIEKMANN, GYLLENBERG, METZ, NAKAOKA E DE ROOS, J. Math. Biol. 61 (2010)].

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1

INTRODUCTION: BACKGROUND AND MOTIVATION

1.1 DELAY EQUATIONS IN MATHEMATICAL MODELING

Ordinary differential equations (ODEs) are one of the classic tools in mathematical modeling and their use is widespread in all areas of science. They describe the evolution in time of a quantity in terms of its derivative at the present time based on the value of the quantity itself only at the present time. However, when trying to describe complex phenomena in a realistic way, ODEs are not always adequate. Indeed, in many phenomena also the past evolution of the quantity influences its future evolution. In other words there may be delayed effects of the history of the phenomenon on the present and future.

One of the most notable areas where delayed effects arise is control theory in engineering [49, 54, 78, 91]. In this context delays may appear as measurement delays, when a certain quantity of interest can be measured only at some distance (in space or time) from the device that affects it. Examples are metal rolling, where the thickness of the plate can be measured only at some distance from the rolls, and the famous problem of the hot shower, where the effect of changing the temperature of the water can be perceived only after a certain amount of time [63]. Delays may also appear as actuation delays, when the effect of actions on the control is not immediate but needs some time to propagate, as is the case of coding, transmission and decoding delays in network-mediated controls [102]. Another famous example of delays in engineering problems is the turning cutter, which can suffer from the regenerative effect: undesired vibrations of the cutter tool and the workpiece make the surface of the workpiece become wavy, hence the cutting force depends on the waviness of the surface of the workpiece generated at the previous rotation [92].

Another classic area where delays appear naturally is mathematical biology [2, 58, 66, 72, 73, 90, 100]. In the context of population dynamics delays can appear due to different stages in the lifespan of individuals (e.g., juveniles and adults, or maturing and mature cells) or to feedback mechanisms (e.g., in networks of neurons). In epidemic models they may be due, e.g., to latent periods (when the individual is infected but is not yet contagious, so that infection is due to interactions with individuals infected at least a certain amount of time in the past) or to incubation periods (when the individual is infected but does not yet have symptoms of the disease).

References to other problems where the history of the phenomenon is important can be found, e.g., in [47].

Delayed effects can be taken into account in differential models by including values of the unknown function at past times, as, e.g., in the simple equation

$$y'(t) = ay(t) + by(t-1). \quad (1.1)$$

The equations thus obtained are called retarded functional differential equations (RFDEs). They are relations of the type

$$y'(t) = G(t, y_t), \quad t \in \mathbb{R},$$

where $G: \mathbb{R} \times Y \rightarrow \mathbb{R}^{d_Y}$ is a function, the state space Y is the space of continuous functions defined on $[-\tau, 0]$ for some $\tau > 0$ (finite delay) or on $(-\infty, 0]$ (infinite delay), and $y_t \in Y$, defined as

$$y_t(\theta) := y(t + \theta), \quad \theta \leq 0, \quad (1.2)$$

is the history or the state at time t . Notice the different domain of y_t depending on the delay being finite or infinite. The function G may involve values of y_t at discrete points (discrete delays) or integrals over some time intervals (distributed delays); the delays may be fixed or they may depend on the current time (time-dependent delays) or on the history (state-dependent delays). In (1.1), which is a linear RFDE with one finite, fixed and discrete delay, the function G is defined as $G(t, \psi) := a\psi(0) + b\psi(-1)$. For a more general linear RFDE with a finite number of finite and fixed delays, both discrete and distributed, see (4.15).

Another kind of equations that take the history of the unknown into account are Volterra functional equations, also called renewal equations (REs). These are delay equations not involving derivatives: they specify the value at present time of the unknown function itself in terms of its past values. REs are relations of the type

$$x(t) = F(t, x_t), \quad t \in \mathbb{R},$$

where $F: \mathbb{R} \times X \rightarrow \mathbb{R}^{d_X}$ is a function, the state space X is the space of L^1 functions defined on $[-\tau, 0]$ for some $\tau > 0$ (finite delay) or on $(-\infty, 0]$ (infinite delay), and $x_t \in X$ is the history or the state at time t , defined as y_t in (1.2).

The name “renewal equations” comes from the study of renewal processes, stochastic models for events occurring randomly in time, where some quantities of interest are described by equations of the type

$$x(t) = \int_0^t k(t - \sigma)x(\sigma) d\sigma + f(t), \quad t \geq 0,$$

which are Volterra integral equations of the second kind. These processes are used to describe, e.g., the replacement of light bulbs, or of machines failing due to wear, or the arrival of customers at a counter [48, 84]. In the context of population dynamics, REs, in the broader sense we use here, arise in particular from the age structure of the population or its physiological structure (e.g., based on the size of individuals) [9, 34, 37, 39, 57, 58, 62].

In constructing equations for population models, some physiological or environmental traits may be better described with RFDEs, while others with REs. As a result, models may involve coupled RFDEs and REs [34, 36, 38,

71, 77]. A famous coupled RE/RFDE model is the size-structured model for the *Daphnia magna* water flea feeding on algae [38]:

$$\begin{cases} b(t) = \int_{a_{\text{repr}}(S_t)}^{a_{\text{max}}} \beta(X(a; S_t), S(t)) \mathcal{F}(a; S_t) b(t-a) da, \\ S'(t) = f(S(t)) - \int_0^{a_{\text{max}}} \gamma(X(a; S_t), S(t)) \mathcal{F}(a; S_t) b(t-a) da, \end{cases} \quad (1.3)$$

where for each a the terms $X(a; S_t) := \bar{X}(a)$ and $\mathcal{F}(a; S_t) := \bar{\mathcal{F}}(a)$ are defined by means of the solutions $\bar{X}$ and $\bar{\mathcal{F}}$ of

$$\begin{cases} \bar{X}'(\alpha) = g(\bar{X}(\alpha), S_t(\alpha-a)), & \alpha \in [0, a], \\ \bar{\mathcal{F}}'(\alpha) = -\mu(\bar{X}(\alpha), S_t(\alpha-a)) \bar{\mathcal{F}}(\alpha), & \alpha \in [0, a], \\ \bar{X}(0) = X_{\text{birth}}, \\ \bar{\mathcal{F}}(0) = 1. \end{cases}$$

The first equation in (1.3) defines the birth rate $b(t)$ of the population at time t , while the second equation defines the concentration of food $S(t)$. Each contribution in the integrals is given by individuals of a certain age a : in the equation for b only by adult individuals (i.e., capable of reproducing), while in the equation for S by all individuals (up to the maximum age a_{max}). Observe that the minimum age for reproduction $a_{\text{repr}}(S_t)$ depends on the history of food availability S_t by a condition imposed on the size, i.e., $X(a_{\text{repr}}(S_t); S_t) = x_{\text{repr}}$. The contributions in the integrals are given by the fertility (β) and consumption (γ) rates of individuals of age a (i.e., born at time $t-a$ at the rate $b(t-a)$) that have survived until time t (with probability $\mathcal{F}(a; S_t)$ depending on the age a and on the history of food availability S_t). The birth and consumption rates depend on the currently available food $S(t)$ and on the size $X(a; S_t)$ of the individual, which also depends on its age a and on the food history S_t .

The *Daphnia* model has several complications: the most outstanding are the state-dependent delay $a_{\text{repr}}(S_t)$ in the equation for b and the fact that X and $\mathcal{F}$ are solutions of external ODEs. Since it serves also as a prototype model for size-structured consumer-resource systems, studying the *Daphnia* model is one of the ultimate goals of the research line to which this thesis belongs (see also chapter 9).

1.2 DYNAMICAL SYSTEMS AND STABILITY

Dynamical systems are formalizations of deterministic processes: they describe phenomena characterized by a quantity evolving univocally in time from an initial condition by means of a prescribed law. They are defined as a triple $\{\mathbb{T}, X, \{T(t)\}_{t \in \mathbb{T}}\}$, where the set of times $\mathbb{T} \subset \mathbb{R}$ contains 0 and is closed under addition, the state space X is the set of the possible states of the system (i.e., the possible values of the evolving quantity), and the evolution operators $\{T(t)\}_{t \in \mathbb{T}}$ are operators on X such that $T(0)$ is the identity on X and $T(t+s) = T(t) \circ T(s)$ for all $t, s \in \mathbb{T}$. Given an initial state $x_0 \in X$ at time 0, the evolution operator $T(t)$ is such that $T(t)x_0$ is the state at time t , denoted by x_t .

Provided that the solutions of corresponding initial value problems (IVPs) are unique, ODEs, RFDEs and REs define dynamical systems. Hence mathematical models based on them can be studied in the framework of dynamical systems. The definition of a dynamical system requires the state of the system to be univocally determined by the initial state and the evolution operators. Thus, dynamical systems associated with ODEs have a finite-dimensional state space, while the introduction of delays notoriously requires an infinite-dimensional state space [31], rendering the problem more difficult in the latter case.

Instead of trying to determine specific solutions of the IVP, the theory of dynamical systems focuses on the qualitative long-term behavior of the solutions. The main interest is in studying the properties of certain invariant sets of the dynamical system, i.e., invariant sets for all evolution operators. In particular, the simplest invariants that are usually considered are constant solutions, also called equilibria or steady states, and periodic solutions, also called cycles. Assume that X is a complete metric space. An invariant set $S \subset X$ is called (Lyapunov) stable if for any neighborhood U of S there exists a neighborhood V of S such that all evolution operators send V in U ; it is unstable otherwise. If S is stable and in addition there exists a neighborhood V_S of S such that $T(t)V_S \rightarrow S$ as $t \rightarrow +\infty$, then S is called asymptotically stable. Observe that in general the stability of an invariant set is a local property, as it regards the behavior of perturbations of the set.

In many applications there is a strong interest in determining the stability properties of invariants. An example from engineering concerns mechanical vibrations, as in the turning cutter mentioned above: it is desirable for the equilibria of the system, representing steady cutting, to be stable in order to ensure that the surface of the workpiece does not become wavy. In studying an epidemic model, an important question is whether an emerging epidemic would result eventually in a disease-free state or in an endemic state, with or without fluctuations: this translates into studying the stability of equilibria and periodic solutions. A concrete example from population dynamics with comments on the biological meaning of the mathematical findings is elaborated in [section 1.4](#).

The local stability properties of a solution of a nonlinear system are related to those of the null solution of the system linearized around that solution, thanks to the principle of linearized stability. If the chosen solution is an equilibrium, the linearized system is autonomous; if instead the solution is periodic, the linearized system has periodic coefficients.

For linear autonomous ODEs, it is well known that the stability of the null solution is determined by the spectrum of the matrix defining the linear system (see, e.g., [55, 67]): if the real part of its rightmost eigenvalue is negative then the null solution is asymptotically stable, if it is positive then the null solution is unstable. For linear autonomous RFDEs and REs, the stability of the null solution is determined by the spectrum of the semigroup of solution operators or, equivalently, by that of its infinitesimal generator [34, 40, 56]: similarly to ODEs, if the real part of the rightmost eigenvalue of the infinitesimal generator is negative then the null solution is asymptotically stable, if it is positive then the null solution is unstable.

For linear periodic RFDEs, as for ODEs, the stability of the null solution is linked by Floquet theory to the monodromy operator, i.e., the evolution operator that shifts the state along the solution by one period (see [40, chapter XIV] and [56, chapter 8]), and in particular to its eigenvalues, called characteristic multipliers. If all multipliers (except possibly the multiplier 1 if it is simple, see subsection 3.1.4) are in the interior of the unit circle then the null solution of the linear system is asymptotically stable, if any of them are outside then the null solution is unstable. For REs, a similar theory is missing. A possible extension in view of the application of sun-star calculus to REs in [34], along with a corresponding principle of linearized stability, is discussed in chapter 3.

For generic linear nonautonomous equations, the stability of the null solution is related to Lyapunov exponents. Investigating the theoretical and computational aspects of Lyapunov exponents is out of the scope of this thesis. Nevertheless, the possibility of employing the discretization techniques developed in chapters 5 and 6 to compute Lyapunov exponents for REs and coupled equations, similarly to how the discretization approach of chapter 4 and [15] is used in [17] for RFDEs, is briefly discussed in chapter 9.

In the case of nonautonomous equations the evolution law changes with time, hence the evolution operators defining the corresponding dynamical systems depend in general on two parameters. The semigroup properties change accordingly, as well. More precisely, the evolution operators $\{T(t, s)\}_{s < t \in \mathbb{T}}$ are operators on X such that $T(t, t)$ is the identity on X for all $t \in \mathbb{T}$ and $T(t, s) = T(t, r) \circ T(r, s)$ for all $s < r < t \in \mathbb{T}$. When $\mathbb{T} = \mathbb{R}$, to emphasize the fact that evolution operators are defined for $s < t$, we often denote them as $T(s + h, s)$ with $s \in \mathbb{R}$ and $h \geq 0$.

The main objective of this thesis is to provide a numerical method to approximate the eigenvalues of a generic evolution operator $T(s + h, s)$, in order to determine the stability of solutions of (nonlinear) REs and coupled equations, extending the ideas of [15] for RFDEs. This can be applied to study the stability of equilibria, by studying the spectrum of the operator $T(h, 0)$ of the autonomous linearized system for any $h > 0$; indeed, the characteristic multipliers μ are related to the characteristic roots λ (i.e., the eigenvalues of the infinitesimal generator) by $\mu = e^{\lambda h}$. Moreover, the method can be applied to study the stability of periodic solutions, by studying the spectrum of the operator $T(s + \Omega, s)$ of the Ω -periodic linearized system.

1.3 NUMERICAL METHODS FOR DELAY EQUATIONS

Delay equations are rather difficult to study analytically and few results can be obtained without resorting to a numerical approach. Indeed, various numerical methods to approximate the spectrum of the operators mentioned above have been proposed in recent years (see, e.g., [16] and the references therein). Delay equations are intrinsically infinite-dimensional, thus numerical methods are based on discretizing the relevant operators into finite-dimensional ones and computing the eigenvalues of the latter.

Observe that all the methods mentioned in this section, including those developed in this thesis, concern equations with finite delay.

For equilibria of RFDEs, in [14] the infinitesimal generator is discretized via pseudospectral differentiation (i.e., exact differentiation of interpolating polynomials, see [95]), while the method of [45] exploits linear multistep methods to discretize the solution operator. For equilibria of REs and coupled REs/RFDEs, the collocation methods of [12, 13] extend the mentioned pseudospectral differentiation approach.

For periodic solutions of RFDEs, DDE-BIFTOOL* [89] constitutes probably the most widespread software for the bifurcation analysis of delay problems (namely RFDEs with constant or state-dependent discrete delays). Its approach consists in computing periodic solutions by piecewise collocation [43, 44], hence providing also a discretization of the monodromy operator. An interesting review of the approaches available in the literature can be found in [61]. Among them, we mention the semi-discretization method [60] and Chebyshev-based collocations [23–25]. While the mentioned methods are limited to special forms of the RFDE with respect to the number and type of delays, the collocation approach of [15] is perhaps the most general, allowing to treat generic evolution operators $T(s + h, s)$ for equations involving any (finite) combination of discrete and distributed delay terms. In particular, note that in the periodic case the method can be applied to monodromy operators for problems with any ratio between the delay and the period, even irrational. This method is presented again in this thesis in chapter 4 for the reader's convenience, with the important change of employing absolutely continuous functions instead of Lipschitz continuous functions in order to prove the convergence of the method, thus relaxing some of the required hypotheses on the coefficients of the equation.

Turning the attention to REs, their importance in population dynamics (see the references in section 1.1 and also [21, 59, 69, 101]) and the lack of methods to approximate the spectra of their evolution operators motivate the research pursued in this thesis. Indeed, the method provided in chapter 5, inspired by the pseudospectral collocation approach of [15], is the first available method for this problem. Due to the different kind of equations (RFDEs describe the derivative of the unknown function, while REs describe the function itself), a fundamental modification with respect to [15] is introduced in the reformulation of the evolution operators. Together with the necessarily different state space, this motivates a complete revisit of [15].

Chapter 5 presents the method in detail, along with a rigorous error analysis and proof of convergence. Moreover, the method is further extended also to coupled REs/RFDEs in chapter 6.

The methods of chapters 4, 5 and 6 allow to construct a matrix approximating a generic evolution operator $T(s + h, s)$. With reasonable hypotheses on the regularity of the coefficients of the equation, the eigenvalues of the matrix are proved to converge to the exact ones, possibly with infinite order. This infinite order of convergence is typical of pseudospectral methods [95] and is especially important, since it ensures a good accuracy with relatively low matrix dimensions and, consequently, low computational cost (e.g., al-

* <http://ddebiftool.sourceforge.net/>

lowing to affordably perform robust analyses of stability and bifurcations, where the same operations need to be repeated numerous times varying the parameters).

The method of [chapter 5](#) has been applied for the first time by the author and colleagues to a special class of REs in the recent work [\[11\]](#), where it was used in comparison with the approach for nonlinear problems described in [\[10\]](#). This and other numerical tests are presented in [chapter 8](#).

For completeness, the literature on delay equations abounds of numerical methods for time integration.

For IVPs, the monographs [\[4\]](#) and [\[20\]](#) and the references therein may serve as a starting point. The former is devoted to methods for RFDEs and delay differential equations of neutral type, including time- and state-dependent delays, based mainly on continuous methods for ODEs, and in particular Runge–Kutta methods (the paper [\[3\]](#) contains a different unifying approach to continuous Runge–Kutta methods, including some more recent results). The latter monograph discusses collocation methods for a large class of Volterra integral and functional equations, including RFDEs and REs.

The cited works contain references also to methods for boundary value problems (BVPs). The recent paper [\[76\]](#) presents a general abstract framework for BVPs for a large class of neutral functional differential equations, applied in [\[74, 75\]](#) to collocation methods. The paper contains also an extensive review of the relevant literature.

Although the philosophy of the present work is to study the stability properties by approximating the spectra of relevant operators without computing the solutions, solving IVPs and BVPs is quite important in the context of the method presented here.

In fact, the adopted approach requires to linearize the delay equation around solutions, which, apart from equilibria, are usually not known explicitly. Hence, it is often required to numerically compute them by solving suitable IVPs in general, or BVPs in the periodic case. Then the equation needs to be linearized around the numerically provided solution and the relevant method can be applied to the resulting linear equation. As mentioned above, in the case of generic nonconstant solutions the stability properties are related to Lyapunov exponents, which are not addressed in this work.

Concerning periodic solutions, which instead are the main focus of the thesis, [\[11, 30\]](#) and [chapter 8](#) contain examples of the application of the pseudospectral collocation method to the monodromy operators of numerically linearized equations. In particular, in [\[30\]](#) and in [section 1.4](#) periodic solutions are computed with the collocation method of [\[43, 70\]](#), while an extension of its ideas to REs is used in [\[11\]](#) and in [section 8.3](#). In [sections 8.1](#) and [8.2](#), instead, periodic solutions are computed by applying the MatCont[†] [\[32\]](#) ODE bifurcation package to the nonlinear ODE systems obtained from the pseudospectral discretization method presented in [\[10\]](#) (also used in [\[11\]](#)).

[†] <https://matcont.sourceforge.io/>

A complete exposition of the problem of reliably computing periodic solutions is out of the scope of this thesis; extending [43] to REs and coupled equation is ongoing work of the author and colleagues (see also [chapter 9](#)).

1.4 AN EXAMPLE FROM MATHEMATICAL BIOLOGY

The material in this section is taken from [30], where the author and colleagues studied a population model with two host species and one parasitoid species. We propose it again here both as an example of biological questions that can be investigated in the framework of delayed dynamical systems and to underline the importance of studying periodic dynamics, since they can be significantly different from the dynamics of equilibria, and the fact that a numerical approach to these problems is often essential.

1.4.1 The biological context

Fruit flies are a major threat to fruit farming. The recent invasion from Eastern Asia of the spotted-wing fruit fly *Drosophila suzukii* into Europe and North America is a cause of great concern for the damages to crops and consequent economic losses [26, 29, 51, 99]. Since controlling this invasive pest with insecticides is problematic and the risk of leaving significant residues on fruit is high, there is a strong interest in strategies for biological control of pests [98]. A common tool consists in employing parasitoids to keep the infestation under control.

Parasitoids are organisms that lay eggs on or in a host organism; the developing larvae feed on the host, ultimately killing it [50]. Many parasitoids are capable of attacking different host species, which in turn are often attacked by different parasitoids. This motivates the interest in models of multi-host-multi-parasitoid interactions.

Introducing in Europe and North America native parasitoids of *Drosophila suzukii* would require careful studies, since it could pose a significant threat to the ecosystem. Fortunately, *Drosophila suzukii* is attacked by several indigenous parasitoids of other *Drosophila* species, such as *Drosophila melanogaster*, the common fruit fly [85]. Some examples are *Leptopilina heterotoma*, which attacks the larval stage, and *Pachycrepoideus vindemiae*, which attacks the pupal stage. These indigenous parasitoids could be employed as biological tools for pest control without posing a significant ecological risk.

It is therefore important to understand the population dynamics of multi-host-multi-parasitoid systems, in order to provide insights to guide the development of biological pest control strategies directed towards the exotic fruit fly.

1.4.2 A two-host-one-parasitoid model

The two-hosts-one-parasitoid model introduced in [30], which is based on [19, 80], makes the following assumptions:

- the life cycle of hosts can be divided into three developmental stages (eggs, larvae and adults) of fixed length;
- the two hosts do not compete;
- intraspecific competition is present only at the larval stage as a density-dependent mortality;
- parasitoids attack only the larval stage of hosts by laying a single egg inside the host;
- juvenile parasitoids emerge from the host larvae after a fixed host-dependent time.

The model is given by

$$\begin{cases} E'_i(t) = R_{E_i}(t) - M_{E_i}(t) - d_{E_i}E_i(t) \\ L'_i(t) = M_{E_i}(t) - M_{L_i}(t) - \alpha_i P(t)L_i(t) - d_{L_i}(L_i(t))L_i(t) \\ A'_i(t) = M_{L_i}(t) - d_{A_i}A_i(t) \\ P'(t) = \sum_{i=1}^2 \alpha_i P(t - T_{P,i})L_i(t - T_{P,i})s_i - d_P P(t) \end{cases} \quad (1.4)$$

where

$$\begin{aligned} R_{E_i}(t) &= \rho_i d_{A_i} A_i(t), \\ M_{E_i}(t) &= \rho_i d_{A_i} A_i(t - T_{E_i}) e^{-d_{E_i} T_{E_i}}, \\ M_{L_i}(t) &= M_{E_i}(t - T_{L_i}) e^{-\int_{t-T_{L_i}}^t (\alpha_i P(y) + d_{L_i}(L_i(y))) dy}, \end{aligned}$$

with $i \in \{1, 2\}$.

It is a system of seven RFDEs with six finite discrete delays (T_{E_i} , $T_{E_i} + T_{L_i}$, $T_{P,i}$) and two finite distributed delays (T_{L_i}).

E_i , L_i , A_i and P are, respectively, the densities of eggs, larvae and adults of host i and of the (adult) parasitoids. M_{E_i} is the maturation rate from eggs to larvae of host i , i.e., it represents the eggs of host i laid time T_{E_i} before t that survived egg mortality. Similarly, $M_{L_i}(t)$ is the maturation rate from larvae to adults of host i , i.e., it represents the eggs of host i matured time T_{L_i} before t that survived both natural mortality and attack by parasitoids.

$d_{L_i}(L_i(t))$ is the mortality of larvae of host i , with $d_{L_i}(L) := \mu_{L_i} + \nu_{L_i}L$, where μ_{L_i} is a constant background mortality and ν_{L_i} is the amount of change in the per capita mortality by adding a new individual. Table 1.1 describes the other parameters of the model.

The quantity $\rho_i d_{A_i}$ represents the birth rate of host i . Indeed, d_{A_i} is the reciprocal of the duration of the adult stage of host i , while ρ_i is the average number of eggs produced in the lifetime of adults.

1.4.3 From the biological to the mathematical questions

Given the biological context described in subsection 1.4.1 and the model (1.4), the key questions biologists ask themselves are the following:

- is it possible for the indigenous and the invading *Drosophila* species to coexist or one will make the other become extinct?

Symbol	Description
ρ_i	average total fecundity of adults of host i
d_{E_i}	mortality of eggs of host i
d_{A_i}	mortality of adults of host i
d_P	mortality of adult parasitoids
α_i	attack rate of adult parasitoids on larvae of host i
s_i	survival of juvenile parasitoids in larvae of host i
T_{E_i}	duration of egg stage of host i
T_{L_i}	duration of larva stage of host i
$T_{P,i}$	duration of juvenile parasitoid stage in host i

Table 1.1: Parameters of model (1.4), $i \in \{1, 2\}$.

- does this depend on parameters that may be controlled in order to influence the evolution of the ecosystem?

Reality is much more complex than the proposed model. Indeed, both *Drosophila melanogaster* and *Drosophila suzukii* are attacked by various species of parasitoids, some of which attack larvae, while others attack the pupal stage, hence a realistic model would require adding more parasitoid species and more life stages of the hosts. Moreover, seasonal changes in the environment (climate, resource availability) would influence the demographic parameters; the attack rate of parasitoids may be density-dependent; consumer species adaptively adjust their behavior in presence of many host species. Nevertheless, this simplified model could provide some insight into the population dynamics of such an ecosystem that might prove useful in modeling biological control strategies based on parasitoids of the indigenous *Drosophila melanogaster*.

Trying to answer the questions above, we proceed from simple to complex. The first aim is to study the equilibria of (1.4), i.e., solutions with constant population densities. We are interested in studying how the model parameters influence the existence (also in the biological sense of being nonnegative) and the stability of the equilibria, in particular of equilibria where the parasitoid and one or both host species coexist. This problem is simple enough to be studied by analytical means. Then we will study the same problems regarding periodic solutions. Unlike equilibria, as observed earlier, it is generally impossible to study by analytical means the existence and stability of periodic solutions of delay equations and to obtain analytical expressions for them, hence it becomes necessary to employ numerical methods.

ANALYTICAL RESULTS ON EQUILIBRIA. Assuming that all parameters are positive, the model (1.4) has seven equilibria, four of which do not present the parasitoid, hence are not interesting in our context. The remaining three equilibria present, respectively, only one host species and the parasitoid, and all three species. Denote them respectively as

$$\begin{aligned}\Gamma_1 &= (\bar{E}_1, \bar{L}_1, \bar{A}_1, 0, 0, 0, \bar{P}_1), \\ \Gamma_2 &= (0, 0, 0, \bar{E}_2, \bar{L}_2, \bar{A}_2, \bar{P}_2), \\ \Gamma &= (E_1^*, L_1^*, A_1^*, E_2^*, L_2^*, A_2^*, P^*).\end{aligned}$$

The equilibria do not necessarily exist in the biological sense, i.e., they are not necessarily nonnegative. In particular, for equilibrium Γ_i , if $\bar{P}_i > 0$, then also $\bar{E}_i$, $\bar{L}_i$ and $\bar{A}_i$ are positive. The value of $\bar{P}_i$ can be computed explicitly, obtaining

$$\bar{P}_i = \frac{\log \rho_i - d_{E_i} T_{E_i} - (\mu_{L_i} + \nu_{L_i} \frac{d_P}{\alpha_i s_i}) T_{L_i}}{\alpha_i T_{L_i}}.$$

By linearizing (1.4) around Γ_1 and Γ_2 and studying the corresponding characteristic equations, the following results are obtained in [30] by analytical means.

Theorem 1.1. *The equilibria Γ_1 and Γ_2 are both unstable if and only if*

$$\begin{cases} \bar{P}_1 < \bar{P}_2 + \frac{\nu_{L_2}}{\alpha_2} \bar{L}_2, \\ \bar{P}_2 < \bar{P}_1 + \frac{\nu_{L_1}}{\alpha_1} \bar{L}_1. \end{cases} \quad (1.5)$$

Corollary 1.2. *If $\nu_{L_1} = \nu_{L_2} = 0$, it is impossible to have mutual invasibility of Γ_1 and Γ_2 .*

Theorem 1.3. *Equilibrium Γ exists (in the biological sense of being nonnegative) if and only if conditions (1.5) hold.*

Recall that, roughly speaking, an equilibrium is unstable if slight perturbations drive the system away from the equilibrium. Thus, the biological meaning of Theorem 1.1 is that if the conditions (1.5) are satisfied and a small number of individuals of one host species is introduced in an environment where only the other host and the parasitoid are present at constant densities, then the newly introduced host will invade the environment (but will not replace the indigenous host species). Corollary 1.2 asserts that the density dependence of the mortality of larvae of at least one host species is essential for this to be possible. Moreover, Theorem 1.3 shows that if both Γ_1 and Γ_2 are unstable, i.e., mutual invasibility of the equilibria occurs, then there exists (in the biological sense) an equilibrium Γ in which the parasitoid and both host species coexist.

Thus we obtained conditions for the coexistence of the hosts at equilibrium and, conversely, for the extinction of the invading species (which corresponds, e.g., to Γ_1 being stable).

PERIODIC SOLUTIONS: NUMERICS COME INTO PLAY. It is well known that systems of predator–prey type can have periodic solutions, especially if they involve delays. The conditions for coexistence may very well change in the periodic regime, hence the previous analysis is irrelevant when trying to answer the same questions looking at cycles instead of equilibria.

As already noted, it is generally impossible to tackle periodic problems for delay equations by analytical means. The recent literature on RFDEs provides efficient numerical methods to perform the several tasks required in this analysis. In particular the methods used in [30] are the following:

- (M1) the method in [14] to approximate the rightmost eigenvalue(s) of the linearization around given equilibria;

Parameter	d_{E_i}	μ_{L_i}	ν_{L_i}	d_P	α_i	s_i	T_{E_i}	T_{L_i}	$T_{P,i}$
Value	0.2	0.1	0.1	1.1	1	1	1	1	1

Table 1.2: Parameter values for the computations, $i \in \{1, 2\}$.

- (M2) the method in [15] to approximate the dominant multiplier(s) of the linearization around given periodic orbits;
- (M3) (an adaptation of) the method in [43] to compute periodic solutions of nonlinear problems;
- (M4) the MATLAB built-in function dde23 [88] to integrate in time Cauchy problems for nonlinear equations.

The four methods are often combined in a framework of parameter continuation, i.e., results for a certain parameter value are obtained starting from results previously computed for a different but close parameter value. For a comprehensive presentation of numerical continuation methods, see [1, 52].

However, the problem remains rather complicated due to the high number of parameters. Hence, to further simplify the analysis, all the parameters are kept fixed at the values listed in Table 1.2, identical between the two hosts, except for those related to the fecundity and adult mortality of the two hosts (ρ_i and d_{A_i}).

The first goal is to determine when single-host periodic solutions exist. Without loss of generality we consider host 1. Fixing $\rho_1 = 5$ and applying method (M1) to the problem linearized around Γ_1 , we determine that the equilibrium Γ_1 is asymptotically stable for $d_{A_1} < d_{HB_1}$, with $d_{HB_1} \approx 0.3089$. At this value, the rightmost complex-conjugate pair of eigenvalues associated with the linearized problem crosses the imaginary axis from left to right, i.e., a Hopf bifurcation occurs and as the equilibrium loses its stability a periodic solution arises.

The periodic solutions are computed with method (M3), using the equilibrium as the initial guess for d_{A_1} slightly above d_{HB_1} , while using the nearest previously computed periodic solution as the initial guess for higher values of d_{A_1} . In particular, a solution $(E_1^\dagger, L_1^\dagger, A_1^\dagger, P_1^\dagger)$ of period $\Omega \approx 18.5938$ is found for $d_{A_1} = 0.35$. Method (M4) was used to confirm the solution, as integrating (1.4) forward in time using the periodic solution as initial value leaves the solution unchanged. The eigenvalues at the Hopf bifurcation and the periodic solution for $d_{A_1} = 0.35$ are shown in Figure 1.1.

For the next aim, consider the periodic solution

$$\Theta_1 := (E_1^\dagger, L_1^\dagger, A_1^\dagger, 0, 0, 0, P_1^\dagger).$$

Linearization of (1.4) around Θ_1 yields a system of linear nonautonomous RFDEs with periodic coefficients, which can be reduced to the single equation

$$A_2'(t) = \rho_2 d_{A_2} A_2(t - T_{E_2} - T_{L_2}) e^{-d_{E_2} T_{E_2} - \mu_{L_2} T_{L_2} - \alpha_2 \int_{t-T_{L_2}}^t P_1^\dagger(y) dy} - d_{A_2} A_2(t).$$

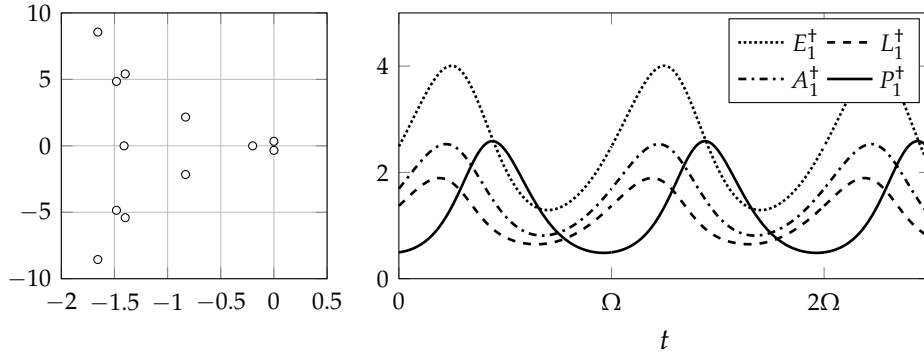


Figure 1.1: Eigenvalues at Hopf bifurcation for $d_{A_1} = d_{HB_1} \approx 0.3089$ and $\rho_1 = 5$ (left) and periodic solution for $d_{A_1} = 0.35$ and $\rho_1 = 5$ (right) of host 1. All other parameter values are as in Table 1.2.

According to Floquet theory (see section 1.2 and chapter 3), the stability of its null solution, and hence the local stability of Θ_1 , depends on whether the dominant multiplier lies inside or outside the unit circle in the complex plane. Thus we keep $\rho_1 = 5$ and $d_{A_1} = 0.35$ fixed and we compute the modulus of this quantity by using method (M2) for varying ρ_2 and d_{A_2} , constructing a surface $\mathbb{R}^2 \rightarrow \mathbb{R}$ whose curve of level 1 divides the (d_{A_2}, ρ_2) -plane into stable and unstable regions.

If Θ_1 is unstable, then a small increase in the densities of host 2 moves away from Θ_1 , i.e., host 2 does not necessarily become extinct and may thus invade the ecosystem. By comparing the boundary of the stability regions for Θ_1 with the invasibility conditions (1.5), we conclude that the periodic case makes the invasion of host 2 easier than for equilibria if d_{A_2} is small and harder if d_{A_2} is large. The stability regions for the periodic solution Θ_1 and the equilibrium Γ_1 are represented in Figure 1.2.

The third objective concerns the effect of fluctuations on mutual invasibility and host coexistence. Again, we keep $\rho_1 = 5$ fixed, but we assume that $d_A := d_{A_1} = d_{A_2}$ varies. Recalling that all other parameters are fixed and equal between the two hosts, we denote them without the i subscript. The mutual invasibility region in the (d_A, ρ_2) -plane is determined as follows.

1. Invasion of host-1-only ecosystem by host 2, $d_A < d_{HB_1}$. From the first inequality of (1.5), the equilibrium Γ_1 is unstable, i.e., host 2 can invade, if and only if

$$\rho_2 > \rho_1 e^{-\frac{v_L d_P T_L}{a s}}. \quad (1.6)$$

Observe that this condition does not depend on d_A .

2. Invasion of host-1-only ecosystem by host 2, $d_A > d_{HB_1}$. We determine the boundary of the stability regions of the periodic solution Θ_1 by repeating the procedure described above: for each value of d_A we compute the periodic solution Θ_1 in absence of host 2 and we apply method (M2) to study its local stability. If Θ_1 is unstable, then host 2 can invade.

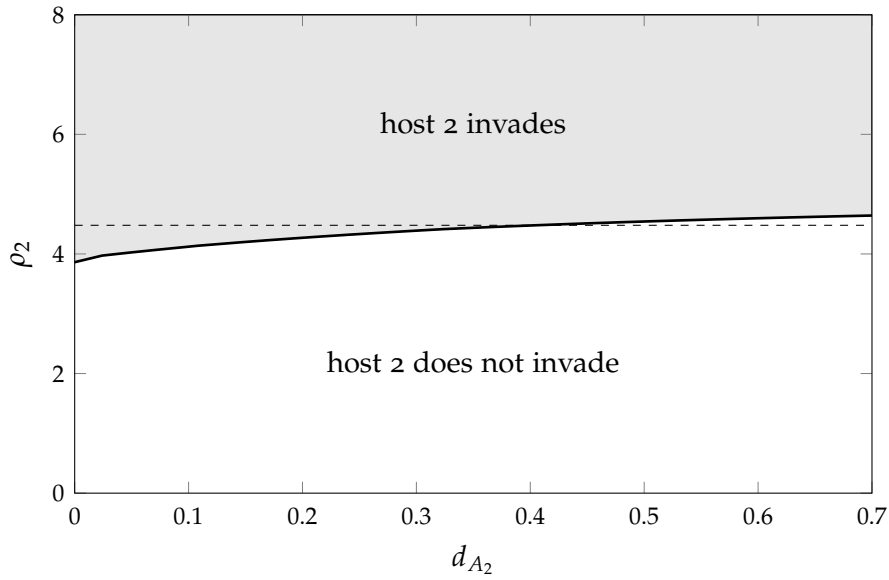


Figure 1.2: Stability regions for the periodic solution Θ_1 : it is stable below, unstable above the thick line. The straight dashed line represents the first of the invasibility conditions (1.5) (see also (1.6)): the equilibrium Γ_1 is stable below, unstable above the line. Parameter values are $\rho_1 = 5$, $d_{A_1} = 0.35$, while the others are from Table 1.2.

3. Invasion of host-2-only ecosystem by host 1, lower d_A . From the second inequality of (1.5), the equilibrium Γ_2 is unstable, i.e., host 1 can invade, if and only if

$$\rho_2 < \rho_1 e^{\frac{v_L d_P T_L}{as}}.$$

Observe that again the condition does not depend on d_A . We consider this stability boundary up to the Hopf bifurcation for Γ_2 , which happens for d_A at some value d_{HB_2} to be determined; more on this in the next step.

4. Invasion of host-2-only ecosystem by host 1, higher d_A . To determine the boundary of the stability regions of the periodic solution Θ_2 in absence of host 1, we repeat the previous reasoning exchanging the role of the two hosts. For each value of d_A and ρ_2 we compute the periodic solution Θ_2 in absence of host 1 and determine its local stability by applying method (M2) to the system (1.4) linearized around Θ_2 . We determine the stability boundary in the (d_A, ρ_2) -plane by selecting the points with dominant multiplier on the unit circle. If Θ_2 is unstable, then host 1 can invade.

Notice that the obtained curve joins the straight line of the previous point for d_A at a value $d_{HB_2} < d_{HB_1}$. Indeed, this is expected, since $\rho_2 > 5$ and hence the Hopf bifurcation for host 2 (in absence of host 1) occurs at a lower value of d_A than that of host 1 (in absence of host 2), which was found for $\rho_1 = 5$.

Observe that what we determined are the boundaries of the stability regions of the single-host notable solutions, switching from the equilibria to the peri-

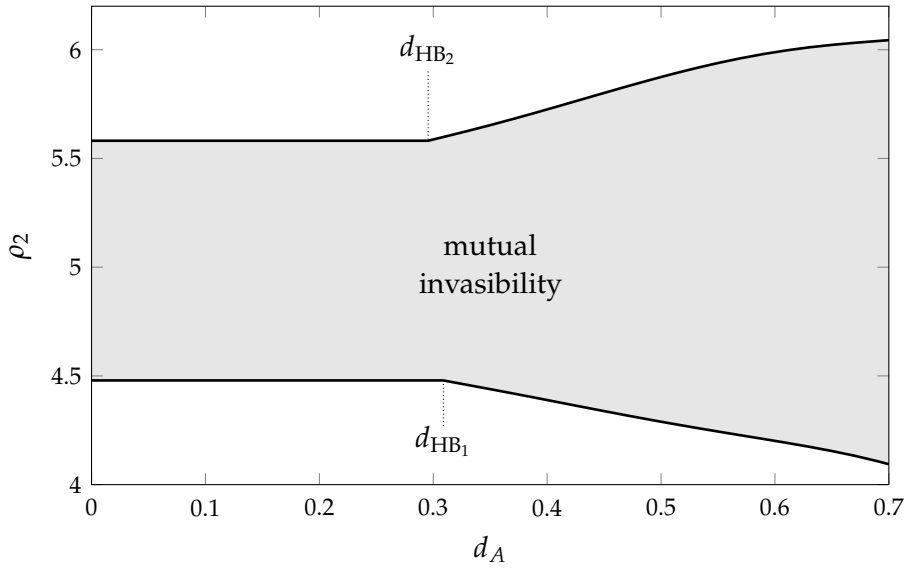


Figure 1.3: Invasibility regions for $d_A := d_{A_1} = d_{A_2}$, $\rho_1 = 5$ and all other parameters as in Table 1.2. For values of (d_A, ρ_2) above the lower curve, the attractor with host 1 coexisting with the parasitoid (Γ_1 or Θ_1) is unstable, hence it can be invaded by host 2. For values below the upper curve, the attractor with host 2 coexisting with the parasitoid (Γ_2 or Θ_2) is unstable, hence it can be invaded by host 1.

odic solutions as the latter appear and exchange the local stability properties with the former. They are represented in Figure 1.3.

We can conclude that as d_A increases, which causes fluctuations in the densities of each host alone with the parasitoid, the mutual invasibility region in (ρ_1, ρ_2) becomes wider, presumably leading to coexistence.

As a final result, the use of this procedure shows that in a periodic regime mutual invasibility is possible also without density dependence of the mortality of larvae (i.e., with $\nu_L = 0$), a case excluded for equilibria by Corollary 1.2. Still, if $d_{A_1} = d_{A_2}$ mutual invasibility is impossible since host 2 can invade a host-1-only periodic solution if and only if $\rho_1 > \rho_2$ and vice versa. Hence, we fix $\nu_{L_1} = \nu_{L_2} = 0$, $\rho_1 = 5$ and $d_{A_1} = 0.3$, we let ρ_2 and d_{A_2} vary, and we proceed as follows.

1. Invasion of host-1-only ecosystem by host 2. We compute the periodic solution Θ_1 for the fixed ρ_1 and d_{A_1} and determine the boundary of its stability regions by applying method (M2) as in the second goal above.
2. Invasion of host-2-only ecosystem by host 1, lower d_A . For values of d_{A_2} below the Hopf bifurcation point there cannot be host-2-only periodic solutions, hence we consider the Γ_2 equilibrium. We determine the boundary of the stability regions from the second inequality of (1.5), which again does not depend on d_{A_2} . If Γ_2 is unstable, then host 1 can invade.
3. Invasion of host-2-only ecosystem by host 1, higher d_A . To determine the boundary of the stability regions of the periodic solution Θ_2 in absence of host 1, we follow the same procedure of exchanging the role of the two hosts. If Θ_2 is unstable, then host 1 can invade. Again,

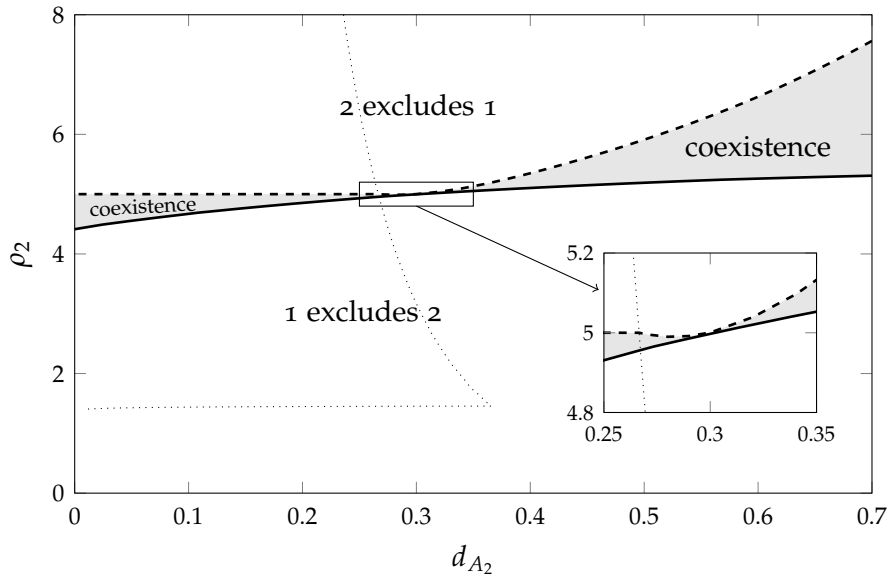


Figure 1.4: Invasibility regions for $\rho_1 = 5$, $d_{A_1} = 0.3$, $v_{L_1} = v_{L_2} = 0$ and all other parameters as in Table 1.2. For values of (d_A, ρ_2) above the solid thick curve, the periodic solution Θ_1 with host 1 coexisting with the parasitoid is unstable, hence it can be invaded by host 2. For values below the dashed thick curve, the attractor with host 2 coexisting with the parasitoid (Γ_2 or Θ_2) is unstable, hence it can be invaded by host 1. The dotted thin curve is the bifurcation curve of the equilibrium Γ_2 and is the union of the locus of Hopf bifurcations (the upper curved part) and of the locus of transcritical bifurcations with the null equilibrium (the lower straight part).

the obtained curve joins the straight line of the previous point at the value of d_{A_2} of the Hopf bifurcation.

The invasibility regions are represented in Figure 1.4.

We conclude that periodicity favors mutual invasibility, and hence possibly host coexistence, even when mortality does not depend on the density.

1.4.4 Conclusions

We can now summarize the results obtained with analytic and numerical means.

- In Theorems 1.1 and 1.3 and Corollary 1.2 we determined conditions for the possibility of coexistence of the host species in presence of the parasitoid (unstable single-host equilibria, existence of single-host equilibria), or for the extinction of the invading species (stable single-host equilibrium).
- We determined conditions for the existence and the stability of nontrivial single-host periodic solutions, concluding that fluctuations make the invasion of the single-host ecosystem easier than for equilibria for a low adult mortality of the invading host and harder for a high mortality (Figure 1.2).

- We determined that fluctuations expand the mutual invasibility region, presumably leading to coexistence (Figure 1.3).
- Finally, fluctuations favor mutual invasibility, and hence possibly host coexistence, even when mortality does not depend on the density (Figure 1.4).

Despite the simplifications made to the model in order to study it effectively, these findings can provide hints to help in modeling biological control strategies based on indigenous parasitoids. Observe however that the numerical approach was essential even in this simplified setting.

Moreover, note that it has been impossible so far to perform similar investigations in the periodic regime when models are described by REs or coupled REs/RFDEs: indeed, this lack motivates the work contained in this thesis.

1.5 ORGANIZATION OF THE THESIS

In this work we are interested in studying the stability properties of periodic solutions of coupled REs/RFDEs. We propose a numerical approach to the problem, along with the relevant theoretical framework.

We collect in chapter 2 some results that are used in the next chapters. In chapter 3 we present Floquet theory and the principle of linearized stability for RFDEs following [40, chapters XII, XIII and XIV], and we discuss its extension to REs in view of [34], which is ongoing work of the author and colleagues. In chapter 4 we present the pseudospectral collocation method for evolution operators of RFDEs [15], with the notable difference of using subspaces of absolutely continuous functions instead of Lipschitz continuous in the proofs; this change is explained at the beginning of the chapter. In chapter 5 the method and the convergence proof are extended to REs, an original result which is the subject of a paper recently submitted for revision, and in chapter 6 to coupled REs/RFDEs; the differences between the methods for the three types of equations are highlighted during the exposition and in chapter 7, which contains some general comments on important elements of the structure of the proofs. Some numerical tests are contained in chapter 8, including an example which the author and colleagues studied also in [11], pairing the method of chapter 5 with that of [10]. Finally, chapter 9 summarizes some open problems and suggests possible future lines of research, while appendix A concludes the thesis with the explicit derivation of the coefficients of the approximated evolution operators for coupled equations and some details on the current MATLAB/Octave implementation.[‡]

Summarizing, the main novel contributions of this work are the preliminary study of the extension to REs of the theoretical framework in chapter 3 and the pseudospectral collocation method for approximating the eigenvalues of generic evolution operators of linear REs (chapter 5) and coupled equations (chapter 6), along with a MATLAB/Octave implementation suitable for all the mentioned classes of delay equations.

[‡] To get the relevant MATLAB/Octave codes, visit the web pages listed on page ii or write to the author.

1.6 PUBLICATIONS

Part of this thesis has already been published by the author and colleagues. In particular, the stability analysis of the two-host-one-parasitoid model of [section 1.4](#) is contained in [\[30\]](#), while the RE with a quadratic nonlinearity of [section 8.3](#) is analyzed in [\[11\]](#), where important analytical results are established and the method of [chapter 5](#) is used along with that of [\[10\]](#) to perform a bifurcation analysis and verify analytical results and conjectures.

Furthermore, the content of [chapter 5](#) and [section 8.1](#) is part of a paper by the author and D. Breda which is currently under revision.

In addition, these topics have been presented by the author and colleagues in various forms and occasions, including talks and posters at scientific conferences and workshops, seminars, and tutorials on the use of the MATLAB/Octave codes for stability and bifurcation analysis.

- [11] D. Breda, O. Diekmann, D. Liessi, and F. Scarabel, *Numerical bifurcation analysis of a class of nonlinear renewal equations*, Electron. J. Qual. Theory Differ. Equ. 65 (2016), pp. 1–24, DOI: [10.14232/ejqtde.2016.1.65](#).
- [30] V. Clamer, A. Pugliese, D. Liessi, and D. Breda, *Host coexistence in a model for two host-one parasitoid interactions*, J. Math. Biol. 75 (2017), pp. 419–441, DOI: [10.1007/s00285-016-1088-z](#).

2

NOTATIONS AND GENERAL RESULTS

This chapter collects notations and general results that are used in different parts of the thesis, in particular some essential results on absolutely continuous functions, approximation of functions, spectra of linear operators, and Volterra integral equations. In the rest of the thesis, these results are explicitly referenced, thus the reader may skip this chapter for the moment and consult it when deemed necessary.

2.1 NOTATIONS AND CONVENTIONS

- $|\cdot|$ denotes any finite-dimensional norm.
- y' denotes the right-hand derivative of a function y .
- 0_U denotes the null element of a linear space U .
- I_U denotes the identity operator on a linear space U . When not ambiguous, e.g., when the identity operator is explicitly applied to an element or when a restriction is explicitly indicated, the subscript U may be omitted.
- Restrictions may not be indicated when not ambiguous, e.g., when the operators are explicitly applied to an element.
- If $X_0, \dots, X_M$ are column vectors, their concatenation is denoted by $(X_0, \dots, X_M)$ in place of the more formal $(X_0^T, \dots, X_M^T)^T$.

The following definition is a standard notation in the theory of delay equations and is used throughout the thesis.

Definition 2.1. For $s \in \mathbb{R}$, $\tau, t_f \geq 0$ and a function g defined on $[s - \tau, s + t_f]$, for each $t \in [s, s + t_f]$ denote with g_t the function defined on $[-\tau, 0]$ as

$$g_t(\theta) := g(t + \theta), \quad \theta \in [-\tau, 0]. \quad (2.1)$$

2.2 ABSOLUTELY CONTINUOUS FUNCTIONS

Absolutely continuous functions have an important role in the convergence proofs of [chapters 4, 5 and 6](#). Indeed, as far as the author is aware, absolute continuity is the least degree of regularity a function is required to have to ensure the uniform convergence of Lagrange interpolation on Chebyshev nodes (see [Theorem 2.17](#) below), which is a key step in all those proofs, albeit in different ways.

Definition 2.2. Let $I := [a, b]$ and $\delta > 0$. A δ -pluri-interval of I is a subset $J \subset I$ such that, for some $K \in \mathbb{N} \setminus \{0\}$,

$$J = \bigcup_{k=1}^K [a_k, b_k],$$

with $a_k < b_k \leq a_{k+1} < b_{k+1}$ for each $k = 1, \dots, K-1$, and

$$\sum_{k=1}^K (b_k - a_k) < \delta.$$

A function $f: I \rightarrow \mathbb{R}$ is *absolutely continuous* on I if, for each $\epsilon > 0$, there exists $\delta > 0$ such that, for each δ -pluri-interval $J = \bigcup_{k=1}^K [a_k, b_k]$,

$$\sum_{k=1}^K |f(b_k) - f(a_k)| < \epsilon.$$

A function $f: I \rightarrow \mathbb{R}^d$ is absolutely continuous on I if all of its components are absolutely continuous on I . Denote the space of absolutely continuous functions on I with values in $\mathbb{R}^d$ as $\text{AC}(I, \mathbb{R}^d)$.

Lemma 2.3 ([86, Theorem 6.11]). *Let $f: I \rightarrow \mathbb{R}^d$. The following are equivalent:*

1. *f is absolutely continuous on I ;*
2. *f has derivative f' almost everywhere in I , the derivative is Lebesgue integrable and, for each $t \in [a, b]$,*

$$f(t) = f(a) + \int_a^t f'(\sigma) \, d\sigma.$$

Definition 2.4. For each $f \in \text{AC}(I, \mathbb{R}^d)$, define the norm

$$\|f\|_{\text{AC}} := \|f\|_1 + \|f'\|_1,$$

where $\|\cdot\|_1$ is the usual norm in $L^1(I, \mathbb{R}^d)$.

Theorem 2.5 ([18, Proposition 8.1]). *$(\text{AC}(I, \mathbb{R}^d), \|\cdot\|_{\text{AC}})$ is a Banach space.*

Theorem 2.6 ([18, Theorem 8.8]). *There exists $C > 0$ such that, for each $f \in \text{AC}(I, \mathbb{R}^d)$,*

$$\|f\|_{\infty} \leq C \|f\|_{\text{AC}},$$

where $\|\cdot\|_{\infty}$ is the usual uniform norm in $C(I, \mathbb{R}^d)$.

The following two propositions are related to the hypotheses imposed on the coefficients of the linear delay equations in order to ensure the convergence of the method. Actually [Proposition 2.7](#) is not related to absolutely continuous functions: it is presented here for the similarity in form and role to [Proposition 2.8](#).

Proposition 2.7. *Let $a < b$, $u \in L^1_{\text{loc}}(\mathbb{R}, \mathbb{R})$ and $C: \mathbb{R} \times [a, b] \rightarrow \mathbb{R}$ such that*

- *for each compact set $K \subset \mathbb{R}$*

$$M_K := \text{ess sup}_{(t, \theta) \in K \times [a, b]} |C(t, \theta)| < +\infty;$$

- the function $t \mapsto C(t, \theta)$ is continuous for almost all $\theta \in [a, b]$, uniformly with respect to θ .

Then the function

$$t \mapsto \int_a^b C(t, \theta) u(t + \theta) d\theta$$

is continuous.

Proof. Let $t_0 \in \mathbb{R}$ and $\epsilon > 0$. For some $\eta > 0$, consider the compact $K := [t_0 + a - \eta, t_0 + b + \eta] \subset \mathbb{R}$. If $u|_K = 0$ the function $t \mapsto \int_a^b C(t, \theta) u(t + \theta) d\theta$ is identically null on $[t_0 - \eta, t_0 + \eta]$, so it is continuous at t_0 . Hence, assume $u|_K \neq 0$.

From the continuity of translation in L^1 there exists $\delta_1 > 0$ such that for all $t_1 \in \mathbb{R}$ if $|t_1 - t_0| < \delta_1$ then $\int_a^b |u(t_1 + \theta) - u(t_0 + \theta)| d\theta < \frac{\epsilon}{2M_K}$. From the assumption on $t \mapsto C(t, \theta)$ there exists $\delta_2 > 0$ such that for all $t_1 \in \mathbb{R}$ and almost all $\theta \in [a, b]$ if $|t_1 - t_0| < \delta_2$ then $|C(t_1, \theta) - C(t_0, \theta)| < \frac{\epsilon}{2\|u|_K\|_1}$, where $\|\cdot\|_1$ is the usual norm in $L^1(K, \mathbb{R})$.

Hence, for all $t_1 \in \mathbb{R}$ if $|t_1 - t_0| < \delta := \min\{\delta_1, \delta_2, \eta\}$ then

$$\begin{aligned} & \left| \int_a^b C(t_1, \theta) u(t_1 + \theta) d\theta - \int_a^b C(t_0, \theta) u(t_0 + \theta) d\theta \right| \\ & \leq \int_a^b |C(t_1, \theta)| |u(t_1 + \theta) - u(t_0 + \theta)| d\theta \\ & \quad + \int_a^b |C(t_1, \theta) - C(t_0, \theta)| |u(t_0 + \theta)| d\theta \\ & < M_K \frac{\epsilon}{2M_K} + \frac{\epsilon}{2\|u|_K\|_1} \int_a^b |u(t_0 + \theta)| d\theta \leq \epsilon, \end{aligned}$$

proving the thesis. $\square$

Proposition 2.8. Let $a < b$, $u \in L^1_{\text{loc}}(\mathbb{R}, \mathbb{R})$ and $C: \mathbb{R} \times [a, b] \rightarrow \mathbb{R}$ such that

- for each compact set $K \subset \mathbb{R}$

$$M_K := \operatorname{ess\,sup}_{(t, \theta) \in K \times [a, b]} |C(t, \theta)| < +\infty;$$

- for each $t \in \mathbb{R}$ and almost all $\theta \in [a, b]$ the function C admits a (finite) directional derivative along the vector $(1, -1)$, denoted $\partial_{(1, -1)} C(t, \theta)$;
- there exist $\bar{\eta} > 0$ and an essentially bounded function $L_{\bar{\eta}}(\theta)$ such that for all η with $0 < |\eta| < \bar{\eta}$ and almost every θ

$$|C(t + \eta, \theta - \eta) - C(t, \theta)| \leq L_{\bar{\eta}}(t + \theta) |\eta|.$$

Then the function

$$t \mapsto \int_a^b C(t, \theta) u(t + \theta) d\theta$$

is absolutely continuous and for almost every $t \in \mathbb{R}$ its derivative is

$$\begin{aligned} \frac{d}{dt} \left(\int_a^b C(t, \theta) u(t + \theta) d\theta \right) &= \int_a^b \partial_{(1, -1)} C(t, \theta) u(t + \theta) d\theta \\ &\quad + C(s + t, b) u(t + b) - C(s + t, a) u(t + a). \end{aligned}$$

Proof. For ease of notation, define $B(t, \sigma) := C(t, \sigma - t)$ and observe that

$$\int_a^b C(t, \theta) u(t + \theta) d\theta = \int_{t+a}^{t+b} C(t, \sigma - t) u(\sigma) d\sigma = \int_{t+a}^{t+b} B(t, \sigma) u(\sigma) d\sigma.$$

The following computations hold:

$$\begin{aligned} & \frac{d}{dt} \left(\int_{t+a}^{t+b} B(t, \sigma) u(\sigma) d\sigma \right) \\ &= \lim_{\eta \rightarrow 0} \frac{1}{\eta} \left[\int_{t+a+\eta}^{t+b+\eta} B(t+\eta, \sigma) u(\sigma) d\sigma - \int_{t+a}^{t+b} B(t, \sigma) u(\sigma) d\sigma \right. \\ & \quad \left. \pm \int_{t+a+\eta}^{t+b+\eta} B(t, \sigma) u(\sigma) d\sigma \right] \\ &= \lim_{\eta \rightarrow 0} \int_{t+a+\eta}^{t+b+\eta} \frac{B(t+\eta, \sigma) - B(t, \sigma)}{\eta} u(\sigma) d\sigma \\ & \quad + \lim_{\eta \rightarrow 0} \frac{1}{\eta} \int_{t+b}^{t+b+\eta} B(t, \sigma) u(\sigma) d\sigma - \lim_{\eta \rightarrow 0} \frac{1}{\eta} \int_{t+a}^{t+a+\eta} B(t, \sigma) u(\sigma) d\sigma \\ &= \lim_{\eta \rightarrow 0} \int_{t+b}^{t+b+\eta} \frac{B(t+\eta, \sigma) - B(t, \sigma)}{\eta} u(\sigma) d\sigma \\ & \quad - \lim_{\eta \rightarrow 0} \int_{t+a}^{t+a+\eta} \frac{B(t+\eta, \sigma) - B(t, \sigma)}{\eta} u(\sigma) d\sigma \\ & \quad + \lim_{\eta \rightarrow 0} \int_{t+a}^{t+b} \frac{B(t+\eta, \sigma) - B(t, \sigma)}{\eta} u(\sigma) d\sigma \\ & \quad + \lim_{\eta \rightarrow 0} \frac{1}{\eta} \int_{t+b}^{t+b+\eta} B(t, \sigma) u(\sigma) d\sigma - \lim_{\eta \rightarrow 0} \frac{1}{\eta} \int_{t+a}^{t+a+\eta} B(t, \sigma) u(\sigma) d\sigma. \end{aligned}$$

By [Theorem 2.28](#), for almost every $t \in \mathbb{R}$,

$$\begin{aligned} \lim_{\eta \rightarrow 0} \frac{1}{\eta} \int_{t+b}^{t+b+\eta} B(t, \sigma) u(\sigma) d\sigma &= B(t, t+b) u(t+b), \\ \lim_{\eta \rightarrow 0} \frac{1}{\eta} \int_{t+a}^{t+a+\eta} B(t, \sigma) u(\sigma) d\sigma &= B(t, t+a) u(t+a). \end{aligned}$$

For all $t \in \mathbb{R}$ and almost every $\sigma \in [t+a, t+b]$ and $0 < |\eta| < \bar{\eta}$,

$$\left| \frac{B(t+\eta, \sigma) - B(t, \sigma)}{\eta} u(\sigma) \right| \leq L_{\bar{\eta}}(\sigma) |u(\sigma)|.$$

Without loss of generality, we can consider only $0 < \eta < \bar{\eta}$, and since $L_{\bar{\eta}}(\sigma)|u(\sigma)|$ is absolutely integrable, by Lebesgue's dominated convergence theorem (Theorem 2.29), for almost every $t \in \mathbb{R}$,

$$\lim_{\eta \rightarrow 0} \int_{t+a}^{t+b} \frac{B(t+\eta, \sigma) - B(t, \sigma)}{\eta} u(\sigma) d\sigma = \int_{t+a}^{t+b} \frac{\partial B}{\partial t}(t, \sigma) u(\sigma) d\sigma.$$

It also follows that

$$\left| \lim_{\eta \rightarrow 0} \int_{t+b}^{t+b+\eta} \frac{B(t+\eta, \sigma) - B(t, \sigma)}{\eta} u(\sigma) d\sigma \right| \leq \lim_{\eta \rightarrow 0} \int_{t+b}^{t+b+\eta} L_{\bar{\eta}}(\sigma) |u(\sigma)| d\sigma = 0,$$

and similarly for the remaining term. Thus

$$\begin{aligned} \frac{d}{dt} \left(\int_{t+a}^{t+b} B(t, \sigma) u(\sigma) d\sigma \right) &= \int_{t+a}^{t+b} \frac{\partial B}{\partial t}(t, \sigma) u(\sigma) d\sigma \\ &\quad + B(t, t+b)u(t+b) - B(t, t+a)u(t+a). \end{aligned}$$

By observing that

$$\frac{\partial B}{\partial t}(t, \sigma) = \partial_{(1,-1)} C(s+t, \sigma-t)$$

the thesis follows. $\square$

Remark 2.9. Observe that if C admits partial derivatives almost everywhere, then

$$\partial_{(1,-1)} C(t, \theta) = \frac{\partial C}{\partial t}(t, \theta) - \frac{\partial C}{\partial \theta}(t, \theta)$$

and

$$\begin{aligned} \frac{d}{dt} \left(\int_a^b C(t, \theta) u(t+\theta) d\theta \right) &= \int_a^b \frac{\partial C}{\partial t}(t, \theta) u(t+\theta) d\theta - \int_a^b \frac{\partial C}{\partial \theta}(t, \theta) u(t+\theta) d\theta \\ &\quad + C(t, b)u(t+b) - C(t, a)u(t+a). \end{aligned}$$

$\triangleleft$

2.3 APPROXIMATION OF FUNCTIONS

Pseudospectral methods are based on discretizing infinite-dimensional problems to finite dimension by algebraic, possibly orthogonal, polynomial collocation [20, 95]. Hence polynomial interpolation plays a major role in these techniques, and thus also in this thesis. In this section we review some standard properties of function approximation, and in particular Lagrange polynomial interpolation, and we collect some key results on the convergence of Lagrange interpolation on Chebyshev nodes.

2.3.1 Best approximation

Let $N \in \mathbb{N}$ and denote with Π_N the space of $\mathbb{R}^d$ -valued polynomials on $[a, b]$ of degree at most N .

Theorem 2.10 ([83, Example I.1]). *Given $f \in C([a, b], \mathbb{R}^d)$, there exists $p_N^* \in \Pi_N$ such that for each $p \in \Pi_N$*

$$\|f - p_N^*\|_\infty \leq \|f - p\|_\infty,$$

where $\|\cdot\|_\infty$ is the uniform norm in $C([a, b], \mathbb{R}^d)$.

Given $f \in C([a, b], \mathbb{R}^d)$, the polynomial p_N^* is the best approximation of f on $[a, b]$ as a polynomial of degree at most N . The best approximation error of f in Π_N is

$$E_N(f) := \|f - p_N^*\|_\infty.$$

The best approximation error is an important part in the estimates of the interpolation error, as is clear from [Theorem 2.14](#).

Define the modulus of continuity of f on $[a, b]$ as

$$\omega(\delta; f) := \sup_{\substack{x_1, x_2 \in [a, b] \\ |x_1 - x_2| \leq \delta}} |f(x_1) - f(x_2)|.$$

Theorem 2.11 (Jackson [83, Corollary 1.4.1]). *If $f \in C([a, b], \mathbb{R}^d)$, then*

$$E_N(f) \leq 6\omega\left(\frac{b-a}{2N}; f\right).$$

Corollary 2.12 ([83, Theorem 1.5]). *If f has a k -th derivative on $[a, b]$ and $N > k$, then*

$$E_N(f) \leq \frac{6^{k+1}e^k}{1+k} \left(\frac{b-a}{2}\right)^k \frac{1}{N^k} \omega\left(\frac{b-a}{2(N-k)}; f^{(k)}\right).$$

2.3.2 Lagrange interpolation

Given a set of distinct points $\{x_n\}_{n \in \{1, \dots, N\}}$ in $[a, b]$ and a set of corresponding values $\{f_n\}_{n \in \{1, \dots, N\}}$, the unique polynomial p of degree at most $N-1$ interpolating the values $\{f_n\}_{n \in \{1, \dots, N\}}$ at the nodes $\{x_n\}_{n \in \{1, \dots, N\}}$ can be written in the Lagrange form as

$$p(x) = \sum_{n=1}^N f_n \ell_n(x),$$

where the Lagrange coefficients ℓ_n are given by

$$\ell_n(x) := \prod_{\substack{i=1 \\ i \neq n}}^N \frac{x - x_i}{x_n - x_i}, \quad n \in \{1, \dots, N\}.$$

The Lebesgue interpolation operator relative to the nodes $\{x_n\}_{n \in \{1, \dots, N\}}$ is the operator $\mathcal{L}_N: C([a, b], \mathbb{R}^d) \rightarrow \Pi_{N-1}$ defined as

$$(\mathcal{L}_N f)(x) := \sum_{n=1}^N f(x_n) \ell_n(x).$$

The Lebesgue constant of the nodes $\{x_n\}_{n \in \{1, \dots, N\}}$ in $[a, b]$ is defined as

$$\Lambda_N := \max_{x \in [a, b]} \sum_{n=1}^N |\ell_n(x)|.$$

Theorem 2.13 ([83, Theorem 4.2]). *For any choice of interpolation nodes*

$$\Lambda_N > \frac{4}{\pi^2} \log N - 1.$$

The following classic theorem provides a bound on the interpolation error, which is exploited in the convergence proofs of [chapters 4, 5 and 6](#). Observe that E_N depends only on f , while Λ_N depends only on the interpolation nodes.

Theorem 2.14 ([83, Theorem 4.1]). *If $f \in C([a, b], \mathbb{R}^d)$, then*

$$\|f - \mathcal{L}_N f\|_\infty \leq (1 + \Lambda_N) E_N(f).$$

2.3.3 Chebyshev nodes

Chebyshev polynomials of the first kind are a family of orthogonal polynomials, defined on $[-1, 1]$ as

$$T_N(x) := \cos(N \arccos(x)), \quad x \in [-1, 1], \quad N \in \mathbb{N}.$$

For $N \in \mathbb{N} \setminus \{0\}$ the zeros of T_N (Chebyshev zeros) are

$$x_{N,n} := \cos\left(\frac{(2n-1)\pi}{2N}\right), \quad n \in \{1, \dots, N\}$$

while its extremal points (Chebyshev extrema) are

$$y_{N,n} := \cos\left(\frac{n\pi}{N}\right), \quad n \in \{0, \dots, N\}. \quad (2.2)$$

[Figure 2.1](#) shows a graphical representation of the construction of Chebyshev zeros and extrema on $[-1, 1]$. Observe that both $\{x_{N,n}\}_{n \in \{1, \dots, N\}}$ and $\{y_{N,n}\}_{n \in \{0, \dots, N\}}$ are sorted from right to left.

Starting from nodes in $[-1, 1]$, corresponding nodes in $[a, b]$ with $a < b$ are defined by the changes of variable

$$\begin{aligned} [-1, 1] \ni s &\mapsto \sigma(s) := \frac{b-a}{2}s + \frac{a+b}{2} \in [a, b], \\ [a, b] \ni \sigma &\mapsto s(\sigma) := \frac{2\sigma - a - b}{b-a} \in [-1, 1], \end{aligned} \quad (2.3)$$

if the order of the nodes needs to be preserved, or

$$\begin{aligned} [-1, 1] \ni s &\mapsto \sigma(s) := \frac{a-b}{2}s + \frac{a+b}{2} \in [a, b], \\ [a, b] \ni \sigma &\mapsto s(\sigma) := \frac{2\sigma - a - b}{a-b} \in [-1, 1], \end{aligned} \quad (2.4)$$

if the order needs to be reversed.

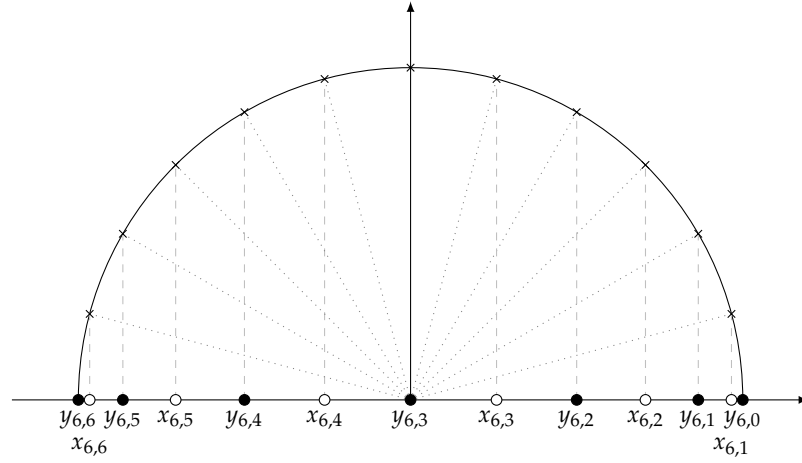


Figure 2.1: Chebyshev zeros (○) and extrema (●) on $[-1, 1]$ for $N = 6$.

Interpolation on Chebyshev nodes has several advantages. In particular, Chebyshev nodes minimize Runge's phenomenon and they constitute an optimal choice, since their Lebesgue constants exhibits a logarithmic growth (compare [Theorems 2.13](#) and [2.15](#)).

Let $N \in \mathbb{N} \setminus \{0\}$ and let $\mathcal{L}_N$ and Λ_N be, respectively, the Lagrange interpolation operator and the Lebesgue constant relative to the Chebyshev zeros on $[-1, 1]$.

Theorem 2.15 ([\[83, Theorem 4.5\]](#)).

$$\Lambda_N < \frac{2}{\pi} \log N + 4.$$

The next theorems are the key results on the convergence of Lagrange interpolation in the convergence proofs of this thesis.

Theorem 2.16 ([\[46, Corollary of Theorem Ia\]](#)). *If $\|\cdot\|_1$ is the usual norm in $L^1([-1, 1], \mathbb{R}^d)$, then*

$$\|\mathcal{L}_N f - f\|_1 \xrightarrow{N \rightarrow +\infty} 0.$$

Theorem 2.17 ([\[65, Theorem 1\]](#)). *If $f(x)$ is absolutely continuous in $(-1, 1)$, then, for $N \rightarrow +\infty$ $(\mathcal{L}_N f)(x)$ converges to $f(x)$ uniformly with respect to x in $(-1, 1)$.*

2.4 LINEAR OPERATORS AND THEIR SPECTRA

The methods developed in this thesis concern the approximation of spectra of evolution operators and the convergence proofs are based on various properties of linear operators on Banach spaces. This section collects several essential results on this topic.

The first result links the invertibility of two operators on different linear spaces, which are related by a pair of operators with the property [\(2.5\)](#). Observe that it does not require the linear spaces to be complete.

Proposition 2.18 ([15, Proposition 3.1], [16, Proposition 6.1]). *Let U and V be linear spaces and let $A: U \rightarrow U$, $P: V \rightarrow U$ and $R: U \rightarrow V$ be linear operators such that*

$$RP = I_V. \quad (2.5)$$

If the operator

$$I_U - PRA: U \rightarrow U, \quad (2.6)$$

is invertible, then the operator

$$I_V - RAP: V \rightarrow V. \quad (2.7)$$

is invertible. Moreover, given $w \in V$, the unique solution $\hat{u} \in U$ of

$$(I_U - PRA)u = Pw \quad (2.8)$$

and the unique solution $\hat{v} \in V$ of

$$(I_V - RAP)v = w \quad (2.9)$$

are related by $\hat{v} = R\hat{u}$ and $\hat{u} = P\hat{v}$.

Proof. If (2.6) is invertible, then, given $w \in V$, (2.8) has a unique solution, say $\hat{u} \in U$. Then

$$\hat{u} = P(RA\hat{u} + w) \quad (2.10)$$

and, by (2.5),

$$R\hat{u} = RA\hat{u} + w \quad (2.11)$$

hold. Hence, by substituting (2.11) in (2.10),

$$\hat{u} = PR\hat{u} \quad (2.12)$$

and, by substituting (2.12) in (2.11), $R\hat{u} = RAPR\hat{u} + w$, i.e., $R\hat{u}$ is a solution of (2.9).

Vice versa, if $\hat{v} \in V$ is a solution of (2.9), then $P\hat{v} = PRAP\hat{v} + Pw$ holds, i.e., $P\hat{v}$ is a solution of (2.8). Hence, by uniqueness, $\hat{u} = P\hat{v}$.

Finally, if $\hat{v}_1, \hat{v}_2 \in V$ are solutions of (2.9), then $P\hat{v}_1 = \hat{u} = P\hat{v}_2$ and, again by (2.5), $\hat{v}_1 = RP\hat{v}_1 = RP\hat{v}_2 = \hat{v}_2$. Therefore $\hat{v} := R\hat{u}$ is the unique solution of (2.9) and the operator (2.7) is invertible. $\square$

The next results are classic theorems on properties of bounded operators, which are used several times in this thesis.

Theorem 2.19 (Bounded inverse theorem [86, Corollary 13.9]). *Let X and Y be Banach spaces and $L: X \rightarrow Y$ a linear, bounded and bijective operator. Then $L^{-1}: Y \rightarrow X$ is bounded.*

Theorem 2.20 (Uniform boundedness theorem, Banach–Steinhaus [86, page 269]). *Let X be a Banach space, Y a normed space and $\mathcal{F}$ a non-empty family of linear and bounded operators from X to Y such that for each $x \in X$ there exists $M_x > 0$ such that, for all $L \in \mathcal{F}$, $\|Lx\|_Y \leq M_x$. Then there exists $M > 0$ such that, for all $L \in \mathcal{F}$, $\|L\|_{Y \leftarrow X} \leq M$. In other words, if $\sup\{\|Lx\|_Y \mid L \in \mathcal{F}\} < +\infty$ for each $x \in X$, then $\sup\{\|L\|_{Y \leftarrow X} \mid L \in \mathcal{F}\} < +\infty$.*

The following theorem is a generalization to Banach spaces of the Banach perturbation lemma (see, e.g., [81, Theorem 2.1.1]).

Theorem 2.21 ([64, Theorem 10.1]). *If U is a Banach space and $A, \tilde{A}$ are linear bounded operators on U such that A is invertible and $\|A^{-1}(\tilde{A} - A)\|_{U \leftarrow U} < 1$, then $\tilde{A}$ has a bounded inverse and*

$$\|\tilde{A}^{-1}\|_{U \leftarrow U} \leq \frac{\|A^{-1}\|_{U \leftarrow U}}{1 - \|A^{-1}(\tilde{A} - A)\|_{U \leftarrow U}}.$$

Let V be a Banach space and $A: V \rightarrow V$ a linear and closed operator. A complex number λ is an eigenvalue of A if there exists $v \in V \setminus \{0_V\}$ such that $(\lambda I_V - A)v = 0$. Such vectors v are the eigenvectors associated with λ . The space spanned by all eigenvectors associated with λ is the eigenspace associated with λ .

The set of all eigenvalues of A is the point spectrum of A , denoted as $\sigma_p(A)$. The spectrum of A , denoted $\sigma(A)$, is the set of all $\lambda \in \mathbb{C}$ such that $\lambda I_V - A$ is not bijective. Obviously $\sigma(A)$ contains $\sigma_p(A)$. By [41, Theorems VII.3.18, VII.4.5 and VII.4.6], if there exists a power of A that is compact, then $\sigma(A) \setminus \{0\} = \sigma_p(A)$; in that case the eigenvalues of A are isolated or they have the single accumulation point 0.

A vector $v \in V$ is a generalized eigenvector associated with λ if there exists $n \in \mathbb{N} \setminus \{0\}$ such that $(\lambda I_V - A)^n v = 0$. The minimum such n is the rank of v . The space spanned by all generalized eigenvectors associated with λ is the generalized eigenspace associated with λ , denoted $\mathcal{E}_\lambda$. The maximum rank l of generalized eigenvectors associated with λ , or equivalently the minimum l such that $(\lambda I_V - A)^l \mathcal{E}_\lambda = \{0_V\}$, is called the ascent of λ .

If v is a generalized eigenvector of rank n associated with λ , the n -tuple of vectors $(v_1, \dots, v_n)$ such that $v_k := (\lambda I_V - A)^{n-k} v$ for $k \in \{1, \dots, n\}$ is a Jordan chain of length n . The vectors satisfy $(\lambda I_V - A)v_{k+1} = v_k$ for $k \in \{1, \dots, n-1\}$ and $(\lambda I_V - A)v_1 = 0$. Observe that $v_n = v$ and v_1 is an eigenvector. The length n of a Jordan chain is a partial multiplicity of λ . It is easy to show that vectors forming a Jordan chain are linearly independent.

Again by [41, Theorems VII.3.18, VII.4.5 and VII.4.6], if there exists a power of A that is compact, then the generalized eigenspace, and hence the eigenspace, associated with an eigenvalue λ have finite dimension. The dimension of the eigenspace is the geometric multiplicity of λ , while the dimension of the generalized eigenspace is the algebraic multiplicity of λ . Observe that the algebraic multiplicity is a sum of partial multiplicities, since a basis of the generalized eigenspace may be composed only of Jordan chains.

The next two propositions show that some pairs of operators share the same spectral properties (possibly excluding those relevant to the eigenvalue 0). In particular, Proposition 2.22 shows that this happens for operators related by (2.5) and (2.13), while Proposition 2.23 shows that the spectral properties are preserved when restricting operators to subspaces, provided that some conditions are fulfilled.

Proposition 2.22 ([15, Proposition 4.1], [16, Lemma 6.1]). *Let U and V be Banach spaces, $A: V \rightarrow V$ a linear and closed operator and $P: V \rightarrow U$ and*

$R: U \rightarrow V$ linear operators such that (2.5) holds. Then A has the same nonzero eigenvalues, with the same geometric and partial multiplicities, of

$$B := PAR: U \rightarrow U. \quad (2.13)$$

Moreover, if $v \in V$ is an eigenvector of A associated with a nonzero eigenvalue λ , then $Pv \in U$ is an eigenvector of B associated with the same eigenvalue λ .

Proof. Assume (2.5) and let $\lambda \in \mathbb{C} \setminus \{0\}$.

If $v \in V$, then $Av = \lambda v$ if and only if $BPv = PARPv = PAv = \lambda Pv$. Vice versa, if $u \in U$, then $Bu = \lambda u$ if and only if $AARu = ARPAr = ARBu = \lambda ARu$. Thus, if v is an eigenvalue of A then $Pv \in U$ is an eigenvalue of B , while if u is an eigenvalue of B then $ARu \in V$ is an eigenvalue of A , and so is $\lambda^{-1}ARu$, and $u = \lambda^{-1}Bu = \lambda^{-1}PARu = P(\lambda^{-1}ARu)$. Hence,

$$P(\ker(\lambda I_V - A)) = \ker(\lambda I_U - B). \quad (2.14)$$

Let $v_1, \dots, v_n \in V$ and $u_1, \dots, u_n \in U$ such that $u_i = Pv_i$ for each $i \in \{1, \dots, n\}$. Then, for $\alpha_1, \dots, \alpha_n \in \mathbb{R}$,

$$\sum_{i=1}^n \alpha_i u_i = P \sum_{i=1}^n \alpha_i v_i = 0 \iff RP \sum_{i=1}^n \alpha_i v_i = \sum_{i=1}^n \alpha_i v_i = 0$$

Thus, if $v_1, \dots, v_n \in \ker(\lambda I_V - A)$ are linearly independent, then for each $i \in \{1, \dots, n\}$ the vectors $u_i := Pv_i$ are in $\ker(\lambda I_U - B)$ and $u_1, \dots, u_n$ are linearly independent. Vice versa, if $u_1, \dots, u_n \in \ker(\lambda I_U - B)$ are linearly independent, there exist $v_1, \dots, v_n \in \ker(\lambda I_V - A)$ such that $u_i = Pv_i$ for each $i \in \{1, \dots, n\}$ and they are linearly independent. Hence,

$$\dim \ker(\lambda I_V - A) = \dim \ker(\lambda I_U - B). \quad (2.15)$$

Equation (2.15) implies that λ is an eigenvalue of A with geometric multiplicity g if and only if it is an eigenvalue of B with the same geometric multiplicity. Equation (2.14) implies the correspondence between the eigenvectors of A and B . It remains to prove that an eigenvalue λ of A and B has the same partial multiplicities with respect to both operators. This is proved by showing a one-to-one correspondence between the Jordan chains of A and B associated with λ .

Let $(v_1, \dots, v_n)$ be a Jordan chain of A associated with an eigenvalue λ . Then $(u_1, \dots, u_n) := (Pv_1, \dots, Pv_n)$ is a Jordan chain of B associated with λ . Indeed, u_1 is an eigenvector of B and for $i \in \{1, \dots, n-1\}$ $(\lambda I_U - B)u_{i+1} = (\lambda I_U - B)Pv_{i+1} = P(\lambda I_V - A)v_{i+1} = Pv_i = u_i$. Vice versa, if $(u_1, \dots, u_n)$ is a Jordan chain of B associated with λ , there exists v_1 eigenvalue of A associated with λ such that $u_1 = Pv_1$. For $i \in \{1, \dots, n-1\}$, if $u_i = Pv_i$ then $(\lambda I_U - B)u_{i+1} = u_i = Pv_i$, which implies $u_{i+1} = P(\lambda^{-1}(v_i + ARu_{i+1}))$. Hence, for $i \in \{1, \dots, n\}$, there exists $v_i \in V$ such that $u_i = Pv_i$ with v_1 an eigenvector of A . For $i \in \{1, \dots, n-1\}$ $v_i = RPv_i = Ru_i = R(\lambda I_U - B)u_{i+1} = R(\lambda I_U - B)Pv_{i+1} = (\lambda I_V - A)v_{i+1}$, thus $(v_1, \dots, v_n)$ is a Jordan chain of A associated with λ . Since P is injective, thanks to (2.5) and the injectivity of I_V , we conclude that the correspondence between Jordan chains of A and B given by P is bijective. $\square$

Proposition 2.23 ([15, Proposition 4.3], [16, Lemma 6.2]). *Let $V \subset U$ be Banach spaces, $A: U \rightarrow U$ a linear and closed operator and $\lambda \in \mathbb{C}$. If*

1. $A(V) \subset V$;
2. *for any $u \in U$ and $v \in V$, $(\lambda I_U - A)u = v$ implies $u \in V$;*

then λ is an eigenvalue of A of geometric multiplicity g if and only if it is an eigenvalue of $A|_V$ of the same geometric multiplicity. Moreover, the eigenvectors associated with λ are the same for A and $A|_V$ and also their partial multiplicities are the same.

Proof. By 1. $A|_V: V \rightarrow V$ and by 2. $\ker(\lambda I_U - A) \subset V$ and $\ker(\lambda I_U - A) = \ker(\lambda I_V - A|_V)$. Hence, λ is an eigenvalue of A with geometric multiplicity g if and only if it is an eigenvalue of $A|_V$ with the same geometric multiplicity, and also the associated eigenvectors are the same. Moreover, by 2. also the Jordan chains associated with λ for A and $A|_V$, and thus the partial multiplicities of λ , are the same. $\square$

Remark 2.24 ([15, Remark 4.4], [16, Remark 6.1]). Observe that for $\lambda \neq 0$ the conditions of Proposition 2.23 are satisfied if $A(U) \subset V$. $\triangleleft$

The following lemma summarizes a useful combination of tools from [27]. It plays an essential role in chapters 4, 5 and 6 in the final step of the proofs that the approximated eigenvalues converge to the exact ones and provides the desired estimate on the convergence order.

Lemma 2.25. *Let U be a Banach space, A a linear and bounded operator on U and $\{A_N\}_{N \in \mathbb{N}}$ a sequence of linear and bounded operators on U such that $\|A_N - A\|_{U \leftarrow U} \rightarrow 0$ for $N \rightarrow +\infty$. If $\mu \in \mathbb{C}$ is an eigenvalue of A with finite algebraic multiplicity v and ascent l , and Δ is a neighborhood of μ such that μ is the only eigenvalue of A in Δ , then there exists a positive integer $\bar{N}$ such that, for any $N \geq \bar{N}$, A_N has in Δ exactly v eigenvalues $\mu_{N,j}$, $j \in \{1, \dots, v\}$, counting their multiplicities. Moreover, by setting $\epsilon_N := \|(A_N - A)|_{\mathcal{E}_\mu}\|_{U \leftarrow \mathcal{E}_\mu}$, where $\mathcal{E}_\mu$ is the generalized eigenspace of μ equipped with the norm $\|\cdot\|_U$ restricted to $\mathcal{E}_\mu$, the following holds:*

$$\max_{j \in \{1, \dots, v\}} |\mu_{N,j} - \mu| = O(\epsilon_N^{1/l}). \quad (2.16)$$

Proof. By [27, Example 3.8 and Theorem 5.22], the norm convergence of A_N to A implies the strongly stable convergence $A_N - \mu I_U \xrightarrow{ss} A - \mu I_U$ for all μ in the resolvent set of A and all isolated eigenvalues μ of finite multiplicity of A . The thesis follows then by [27, Proposition 5.6 and Theorem 6.7]. $\square$

2.5 VOLTERRA INTEGRAL EQUATIONS

This section summarizes some basic facts on Volterra integral equations, following [53, chapter 9].

Let $J \subset \mathbb{R}$ be an interval. A measurable function $K: J^2 \rightarrow \mathbb{C}^{d \times d}$ such that $K(t, s) = 0$ for $s > t$ is a Volterra kernel on J . A Volterra kernel K is of type L^1 on J if

$$\operatorname{ess\,sup}_{s \in J} \int_J |K(t, s)| \, dt < +\infty$$

(see [53, Definition 9.2.2 and Proposition 9.2.7]). If K and R are Volterra kernels of type L^1 on J such that

$$R(t, s) + \int_J K(t, u)R(u, s) du = R(t, s) + \int_J R(t, u)K(u, s) du = K(t, s)$$

on J^2 , then R is a Volterra resolvent of type L^1 of K (see [53, Definitions 9.2.3 and 9.3.1]).

Theorem 2.26 ([53, Theorem 9.3.6]). *If K is a kernel of type L^1 on J that has a resolvent R of type L^1 on J , and if $f \in L^1(J, \mathbb{R}^d)$, then the equation*

$$x(t) = \int_J K(t, s)x(s) ds + f(t), \quad t \in J, \quad (2.17)$$

has a unique solution x in $L^1(J, \mathbb{R}^d)$. This solution is given by the variation of constants formula

$$x(t) = \int_J R(t, s)f(s) ds + f(t), \quad t \in J.$$

Theorem 2.27 ([53, Corollary 9.3.14]). *Let K be a Volterra kernel of type L^1 on J . If J can be divided into finitely many subintervals J_i such that, on each J_i ,*

$$\operatorname{ess\,sup}_{s \in J_i} \int_{J_i} |K(t, s)| dt < 1,$$

then K has a resolvent of type L^1 on J .

Hence, by Theorems 2.26 and 2.27, if the hypothesis of Theorem 2.27 is verified, then (2.17) has a unique solution in L^1 for each $f \in L^1(J, \mathbb{R}^d)$.

2.6 OTHER RESULTS

This final section collects some standard results that are used in the following chapters. They are presented here for the reader's convenience.

Theorem 2.28 (Lebesgue differentiation theorem, second formulation [86, Theorem 6.14]). *Let $f: \mathbb{R} \rightarrow \mathbb{C}$ be an absolutely integrable function. Then, for almost every $x \in \mathbb{R}$,*

$$\begin{aligned} \lim_{h \rightarrow 0^+} \frac{1}{h} \int_x^{x+h} f(t) dt &= f(x), \\ \lim_{h \rightarrow 0^+} \frac{1}{h} \int_{x-h}^x f(t) dt &= f(x). \end{aligned}$$

Theorem 2.29 (Lebesgue's dominated convergence theorem [86, page 376]). *Let X be a measure space with measure μ and $\{f_n\}_{n \in \mathbb{N}}$ a sequence of measurable functions on X for which $f_n \rightarrow f$ as $n \rightarrow \infty$ pointwise almost everywhere on X and the function f is measurable. Assume there is a nonnegative function g that is integrable over X and dominates the sequence $\{f_n\}_{n \in \mathbb{N}}$ in the sense that $|f_n| \leq g$ almost everywhere on X for all $n \in \mathbb{N}$. Then f is integrable over X and*

$$\lim_{n \rightarrow \infty} \int_X f_n d\mu = \int_X f d\mu.$$

Theorem 2.30 (Riesz representation theorem for L^p [86, page 400]). *Let X be a σ -finite measure space with measure μ , $1 \leq p < +\infty$, and q the conjugate of p . For $f \in L^q(X, \mathbb{R}^d)$, define $T_f \in (L^p(X, \mathbb{R}^d))^*$ as*

$$T_f(g) := \int_X fg \, d\mu, \quad g \in L^p(X, \mathbb{R}^d).$$

Then T is an isometric isomorphism of $L^q(X, \mathbb{R}^d)$ onto $(L^p(X, \mathbb{R}^d))^$.*

Theorem 2.31 (Kolmogorov–M. Riesz–Fréchet [18, Theorem 4.26]). *Let $\mathcal{F}$ be a bounded subset of $L^p(\mathbb{R}, \mathbb{R}^d)$ with $1 \leq p < +\infty$. Assume that*

$$\lim_{h \rightarrow 0} \|\tau_h f - f\|_p = 0$$

uniformly in $f \in \mathcal{F}$, i.e., for each $\epsilon > 0$ there exists $\delta > 0$ such that $\|\tau_h f - f\|_p < \epsilon$ for all $f \in \mathcal{F}$ and all $h \in \mathbb{R}$ such that $|h| < \delta$. Then the closure of $\mathcal{F}|_\Omega$ in $L^p(\Omega, \mathbb{R}^d)$ is compact for any measurable set $\Omega \subset \mathbb{R}$ with finite measure.

Here τ_h denotes the translation by h defined by $(\tau_h f)(t) := f(t + h)$ and $\mathcal{F}|_\Omega$ denotes the restrictions to Ω of the functions in $\mathcal{F}$.

Theorem 2.32 (Grönwall's inequality [82, Lemma 1.4.1]). *Suppose that $a < b$, let ψ be a nonnegative function in $L^1(a, b)$, and let α and φ be continuous functions defined on $[a, b]$. Moreover, suppose that α is nondecreasing. If for all $t \in [a, b]$*

$$\varphi(t) \leq \alpha(t) + \int_a^t \psi(s) \varphi(s) \, ds,$$

then for all $t \in [a, b]$

$$\varphi(t) \leq \alpha(t) e^{\int_a^t \psi(s) \, ds}.$$

Theorem 2.33 (Contraction mapping theorem, Banach–Caccioppoli [86, page 216]). *Let (X, d) be a complete metric space and $f: X \rightarrow X$ a contraction, i.e., a Lipschitz continuous function of constant $0 \leq k < 1$. Then f has a unique fixed point in X .*

3

FLOQUET THEORY AND POINCARÉ MAPS

This chapter collects the main results of [40] on Floquet theory and the principle of linearized stability for delay equations, on which our numerical approach to stability is based. For RFDEs the theory is complete, since it is shown in [40] that they fall within the abstract framework developed therein. On the contrary, REs and coupled equations are not covered in [40], thus a contribution of this thesis is an attempt to check whether the abstract theory can be applied to this case. Indeed, in section 3.3 we discuss the validation of the necessary hypotheses, in light of the extension of sun-star calculus to this class of equations [34].

Notice that this study has a preliminary nature and is not complete. In fact, our main focus is on the numerical analysis of the methods proposed in chapters 5 and 6. Nevertheless, section 3.3 provides a considerable part of the required work and gives some valuable hints on the feasibility of applying the abstract theory to REs and coupled equations.

The difference between the notations of this and the following chapters is preserved in order to maintain a notational uniformity, respectively, with [34, 40] and [16]. However, this should not cause misunderstandings.

3.1 ABSTRACT THEORY

For the reader's convenience, this section collects from [40] the main abstract results on sun-star calculus, evolutionary systems for time-dependent linear equations, Floquet theory, and the principle of linearized stability. In particular, subsection 3.1.1 contains material from [40, chapter II], subsection 3.1.2 from [40, chapter XII], subsection 3.1.3 from [40, chapter XIII], and subsection 3.1.4 from [40, chapter XIV]. We omit the proofs and many details, which can be found in the cited monography.

3.1.1 Suns and stars

The idea of sun-star calculus for semigroups of operators consists in considering dual spaces and restricting the relevant semigroups to subspaces where some desired properties, namely being strongly continuous, are preserved.

Let X be a Banach space with norm $\|\cdot\|_X$. The space of continuous linear functionals on X is called the dual space of X and denoted by X^* . For $x \in X$ and $x^* \in X^*$ we write $\langle x^*, x \rangle := x^*(x)$. The dual space is a Banach space when equipped with the norm

$$\|x^*\|_{X^*} = \sup_{\|x\|_X \leq 1} |\langle x^*, x \rangle|.$$

It follows that

$$\|x\|_X = \sup_{\|x^*\|_{X^*} \leq 1} |\langle x^*, x \rangle|.$$

The weak* topology on X^* is the coarsest topology such that for each $x \in X$ the functional $x^* \mapsto \langle x^*, x \rangle$ on X^* is continuous. A sequence $\{x_n^*\}_{n \in \mathbb{N}}$ in X^* converges to $x^* \in X^*$ as $n \rightarrow +\infty$ in the weak* topology if and only if for each $x \in X$ the sequence $\{\langle x_n^*, x \rangle\}_{n \in \mathbb{N}}$ converges to $\langle x^*, x \rangle$ as $n \rightarrow +\infty$.

Let $L: X \rightarrow X$ be a linear bounded operator on X . Its adjoint is the operator $L^*: X^* \rightarrow X^*$ defined by

$$\langle x^*, Lx \rangle = \langle L^*x^*, x \rangle$$

for every $x \in X$ and $x^* \in X^*$. The adjoint operator is a uniquely defined linear and bounded operator on the dual space such that $\|L^*\|_{X^* \leftarrow X^*} = \|L\|_{X \leftarrow X}$.

Let $A: \mathcal{D}(A) \rightarrow X$ be a linear unbounded densely defined operator with $\mathcal{D}(A) \subset X$. Its adjoint is the operator $A^*: \mathcal{D}(A^*) \rightarrow X^*$ defined by

$$\begin{aligned} \mathcal{D}(A^*) &:= \{x^* \in X^* \mid \text{there exists } y^* \in X^* \text{ such that} \\ &\quad \langle x^*, Ax \rangle = \langle y^*, x \rangle \text{ for all } x \in \mathcal{D}(A)\} \end{aligned}$$

and $A^*x^* = y^*$ with y^* given by the condition in the domain.

A family $T = \{T(t)\}_{t \geq 0}$ of linear and bounded operators on X is called a semigroup of operators [28, 40] if

$$\begin{aligned} T(0) &= I_X, \\ T(t)T(s) &= T(t+s), \quad t, s \geq 0. \end{aligned}$$

The semigroup of operators $\{T(t)\}_{t \geq 0}$ is called strongly continuous (or C_0) if for each $x \in X$ the function $\|T(t)x - x\|_X \rightarrow 0$ as $t \downarrow 0$.

The linear operator $A: \mathcal{D}(A) \rightarrow X$ defined as

$$Ax := \lim_{t \downarrow 0} \frac{1}{t} (T(t)x - x),$$

with domain

$$\mathcal{D}(A) := \{x \in X \mid \lim_{t \downarrow 0} \frac{1}{t} (T(t)x - x) \text{ exists}\},$$

is called the infinitesimal generator of T and is in general unbounded. It is a closed densely defined operator.

Let $T = \{T(t)\}_{t \geq 0}$ be a C_0 semigroup on X . The family $T^* := \{T^*(t)\}_{t \geq 0}$, with $T^*(t) := (T(t))^*$, which is a semigroup of operators on X^* , is the adjoint semigroup of T . It is not necessarily strongly continuous. However, for all $x \in X$ and $x^* \in X^*$, the function $t \mapsto \langle T^*(t)x^*, x \rangle$ is continuous, i.e., given $x^* \in X^*$ the orbit $t \rightarrow T^*(t)x^*$ is continuous in the weak* topology.

The adjoint A^* of the infinitesimal generator A is the infinitesimal generator of the adjoint semigroup T^* in the weak* sense, i.e., the limit

$$\lim_{t \downarrow 0} \frac{1}{t} \langle T^*(t)x^* - x^*, x \rangle$$

converges for all $x \in X$ if and only if $x^* \in \mathcal{D}(A^*)$ and in that case it equals $\langle A^*x^*, x \rangle$.

The domain $\mathcal{D}(A^*)$ is weak*-dense but not necessarily norm-dense. Its norm closure is

$$X^\odot := \overline{\mathcal{D}(A^*)} = \{x^* \in X^* \mid \lim_{t \downarrow 0} \|T^*(t)x^* - x^*\|_{X^*} = 0\},$$

i.e., the subspace on which T^* is strongly continuous ($\odot$ is a common symbol for the Sun). Defining $T^\odot = T^*|_{X^\odot} := \{T^*(t)|_{X^\odot}\}_{t \geq 0}$, the generator $A^\odot$ of $T^\odot$ is given by

$$\mathcal{D}(A^\odot) := \{x^\odot \in \mathcal{D}(A^*) \mid A^*x^\odot \in X^\odot\}$$

and $A^\odot x^\odot := A^*x^\odot$.

Applying the same reasoning to the C_0 semigroup $T^\odot$ on $X^\odot$ with generator $A^\odot$ we obtain the dual space $X^{\odot*}$, the adjoint semigroup $T^{\odot*}$ with the adjoint $A^{\odot*}$ of the generator, and the subspace $X^{\odot\odot} := \overline{\mathcal{D}(A^{\odot*})}$ on which $T^{\odot*}$ is strongly continuous and has generator $A^{\odot\odot}$.

We can define an embedding $j: X \rightarrow X^{\odot*}$ as

$$\langle jx, x^\odot \rangle = \langle x^\odot, x \rangle$$

for each $x \in X$ and $x^\odot \in X^\odot$. Observe that $j(X) \subset X^{\odot\odot}$ and $T^{\odot*}(t)j = jT(t)$ for each $t \geq 0$. If $j(X) = X^{\odot\odot}$, X is called sun-reflexive with respect to T .

3.1.2 Evolutionary systems and time-dependent linear equations

Given $\alpha, \omega \in \mathbb{R} \cup \{-\infty, +\infty\}$, with $\alpha < \omega$, consider the set $\Delta \subset \mathbb{R}^2$ defined as

$$\Delta := \{(t, s) \in \mathbb{R}^2 \mid \alpha \leq s \leq t \leq \omega\},$$

where $\leq$ should be read as $<$ whenever one of the sides is infinite.

A family $U = \{U(t, s)\}_{(t, s) \in \Delta}$ of linear and bounded operators on X is called a (forward) evolutionary system (or evolution family) [28, 40] if

$$U(s, s) = I_X, \quad \alpha \leq s \leq \omega, \quad (3.1)$$

$$U(t, r)U(r, s) = U(t, s), \quad \alpha \leq s \leq r \leq t \leq \omega. \quad (3.2)$$

The evolutionary system U is called strongly continuous if for each $x \in X$ the function $\Delta \ni (t, s) \mapsto U(t, s)x \in X$ is continuous.

Theorem 3.1. Let $T_0 = \{T_0(t)\}_{t \geq 0}$ be a C_0 semigroup and let X be sun-reflexive with respect to T_0 . Let $\{B(t)\}_{\alpha \leq t \leq \omega}$ be a strongly continuous family of linear and bounded operators from X to $X^{\odot*}$, i.e., for every $x \in X$ the function $t \mapsto B(t)x$ is continuous from $[\alpha, \omega]$ to $X^{\odot*}$. The variation of constants equation

$$U(t, s)\varphi = T_0(t - s)\varphi + j^{-1}\left(\int_s^t T_0^{\odot*}(t - \sigma)B(\sigma)U(\sigma, s)\varphi \, d\sigma\right), \quad (3.3)$$

where $(t, s) \in \Delta$ and $\varphi \in X$, uniquely defines a strongly continuous forward evolutionary system U . The expansion

$$U(t, s) = \sum_{n=0}^{+\infty} U_n(t, s),$$

where

$$U_n(t, s)\varphi = j^{-1} \left(\int_s^t T_0^{\odot*}(t - \sigma) B(\sigma) U_{n-1}(\sigma, s) \varphi \, d\sigma \right)$$

and $U_0(t, s) = T_0(t - s)$, converges in the uniform (operator) topology, uniformly in Δ . Furthermore,

$$\|U(t, s)\|_{X \leftarrow X} \leq M e^{(\omega_0 + MK(t, s))(t-s)},$$

where

$$K(t, s) = \sup_{s \leq \sigma \leq t} \|B(\sigma)\|_{X^{\odot*} \leftarrow X}$$

and M and ω_0 are such that $\|T_0(t)\|_{X \leftarrow X} \leq M e^{\omega_0 t}$.

Depending on the choice of the semigroup T_0 , the abstract equation (3.3) and the evolutionary system it determines correspond to different kinds of equations, as shown in sections 3.2 and 3.3.

3.1.3 Floquet theory

Floquet theory for linear periodic equations links the eigenvalues of the monodromy operators to the stability of the null solution, by providing a way to rewrite the solution as the product of a periodic part and an exponential part.

Let X be a complex Banach space and let the strongly continuous family $\{B(t)\}_{t \in \mathbb{R}}$ of linear bounded operators from X to $X^{\odot*}$ be periodic, i.e., there exists $p > 0$ such that $B(t + p) = B(t)$ for all $t \in \mathbb{R}$. By Theorem 3.1,

$$u(t) = T_0(t - s)u(s) + j^{-1} \left(\int_s^t T_0^{\odot*}(t - \sigma) B(\sigma) u(\sigma) \, d\sigma \right) \quad (3.4)$$

defines a unique evolutionary system $\{U(t, s)\}_{(t, s) \in \Delta}$ where Δ is defined with $\alpha = -\infty$ and $\omega = +\infty$.

The following proposition shows some important identities related to the periodicity of the evolutionary system.

Proposition 3.2. 1. For $t \geq s$, $U(t + p, s + p) = U(t, s)$.

2. For $t \geq 0$ and $j \in \mathbb{Z}$, $U(t + jp, jp) = U(t, 0)$.

3. For $t \in \mathbb{R}$ and $j \in \mathbb{N}$, $U(t + jp, t) = U(t + p, t)^j$.

Define the family of period maps $\{V_t\}_{t \in \mathbb{R}}$ by $V_t := U(t + p, t)$. Observe that $V_{t+p} = V_t$ for all $t \in \mathbb{R}$.

The next result links the asymptotic stability of the null solution with the eigenvalues of period maps by showing a sufficient condition for which the norm of evolution operators decreases exponentially. As usual, we denote the spectrum of an operator L with $\sigma(L)$. For $r \geq 0$ and $c \in \mathbb{C}$ we write also $B_r(c) := \{x \in \mathbb{C} \mid |x - c| < r\}$, i.e., the open ball in the complex plane centered in c with radius r .

Theorem 3.3. *Let $s \in \mathbb{R}$. Assume*

$$\sigma(V_s) \subset B_1(0).$$

Then there exist $C \geq 0$ and $\epsilon > 0$ such that for all $t \geq s$

$$\|U(t, s)\|_{X \leftarrow X} \leq Ce^{-\epsilon(t-s)}.$$

A bounded linear operator on X has the spectral isolation property if each nonzero point in its spectrum is an isolated point of the spectrum. We assume that each element of the family $\{V_t\}_{t \in \mathbb{R}}$ has the spectral isolation property. This is ensured if, e.g., iterates of V_t are compact.

Define $\sigma_t := \sigma(V_t)$ and, for $\lambda \in \sigma_t \setminus \{0\}$, denote the corresponding generalized eigenspace of V_t by $\mathcal{M}_{\lambda, t}$.

Theorem 3.4. *Let $t \geq s$ and $\lambda \in \mathbb{C} \setminus \{0\}$. Then $\lambda \in \sigma_t$ if and only if $\lambda \in \sigma_s$. In this case, $U(t, s)$ reduces to a topological isomorphism of $\mathcal{M}_{\lambda, s}$ onto $\mathcal{M}_{\lambda, t}$.*

The spectral points $\lambda \in \sigma_t \setminus \{0\}$ for an arbitrary $t \in \mathbb{R}$ are the Floquet multipliers of the evolutionary system. [Theorem 3.4](#) shows that all period maps share the same spectrum, hence the Floquet multipliers are well defined.

Let λ be a Floquet multiplier. For each $t \geq 0$ define the operator

$$\begin{aligned} U_t: \mathcal{M}_{\lambda, 0} &\rightarrow \mathcal{M}_{\lambda, t} \\ \varphi &\mapsto U(t, 0)\varphi. \end{aligned}$$

For each $t \leq 0$ define the operator

$$\begin{aligned} \hat{U}_t: \mathcal{M}_{\lambda, t} &\rightarrow \mathcal{M}_{\lambda, 0} \\ \varphi &\mapsto U(0, t)\varphi. \end{aligned}$$

For $t < 0$ define also $U_t := (\hat{U}_t)^{-1}$. Observe that U_p maps $\mathcal{M}_{\lambda, 0}$ onto $\mathcal{M}_{\lambda, p} = \mathcal{M}_{\lambda, 0}$. Observe also that $\sigma(U_p) = \{\lambda\}$.

Proposition 3.5. *There exists a linear and bounded operator W on $\mathcal{M}_{\lambda, 0}$ such that $U_p = e^{pW}$.*

For each $t \in \mathbb{R}$ define the operator $R_t: \mathcal{M}_{\lambda, 0} \rightarrow X$ as $R_t\varphi := U_t e^{-tW}\varphi$.

Proposition 3.6. *For all $t \in \mathbb{R}$, $R_{t+p} = R_t$.*

The final result of this subsection provides the structure of the orbits in the generalized eigenspace $\mathcal{M}_{\lambda, 0}$: as shown by [Propositions 3.5](#) and [3.6](#), they depend on a periodic operator and on an exponential operator that is linked to the Floquet multiplier λ via the operator U_p , which on $\mathcal{M}_{\lambda, 0}$ coincides with V_0 .

Theorem 3.7. *For every solution $u: \mathbb{R} \rightarrow X$ of (3.4) such that $u(t) \in \mathcal{M}_{\lambda, t}$ for all t , we have $u(t) = R_t e^{tW} u(0)$ for all $t \in \mathbb{R}$.*

3.1.4 Poincaré maps and linearized stability

Now let X be a real Banach space, sun-reflexive with respect to the C_0 semigroup T_0 and consider a C^1 function $G: X \rightarrow X^{\odot*}$.

Let $u: \mathbb{R} \rightarrow X$ be a p -periodic solution of

$$u(t) = T_0(t)\varphi + j^{-1} \left(\int_0^t T_0^{\odot*}(t-\sigma)G(u(\sigma)) \, d\sigma \right) \quad (3.5)$$

for some $\varphi \in X$. According to the results of [subsection 3.1.2](#), the linear equation

$$w(t) = T_0(t-s)w(s) + j^{-1} \left(\int_s^t T_0^{\odot*}(t-\sigma)DG(u(\sigma))w(\sigma) \, d\sigma \right), \quad (3.6)$$

where DG is the Fréchet derivative of G , determines an evolutionary system $\{U_u(t, s)\}_{t \geq s}$.

By complexifying this equation (see [\[40, chapter III\]](#)) and applying the results of [subsection 3.1.3](#), assuming the spectral isolation property for the family $\{U_u(t + p, t)\}_{t \in \mathbb{R}}$, we obtain Floquet multipliers. Indeed the sets $\sigma(U_u(t + p, t)) \setminus \{0\}$ coincide for all $t \in \mathbb{R}$ and all translates of the periodic solution u .

Proposition 3.8. *Suppose u is differentiable. Then 1 is a Floquet multiplier and there exists a translate v of u such that $v'(0) := Dv(0)1$ is an eigenvector of $U_v(p, 0)$ associated with the eigenvalue 1.*

The periodic orbit $u(\mathbb{R})$ is called hyperbolic if 1 is a simple eigenvalue (it has a one-dimensional corresponding generalized eigenspace) and no other Floquet multiplier is on the unit circle.

Let $H \subset X$ be a hyperplane such that $u(0) \in H$. Consider the semiflow Σ , i.e., the map $[0, +\infty) \times X \ni (t, \varphi) \rightarrow u(t) \in X$, where u is the solution of (3.5) (see [\[40, chapter VII\]](#)). Assume that it is C^1 in a neighborhood of $(p, u(0))$ and that $u'(0) = u'(p) = D_1\Sigma(p, u(0))1 \notin T_{u(0)}H$. $D_1\Sigma$ is the Fréchet partial derivative of Σ with respect to the first argument; $T_{u(0)}H$ is the tangent space to H at $u(0)$, which is a closed subspace of X of codimension 1.

The next proposition provides a map which gives the time when the semiflow starting from a neighborhood of $u(0)$ returns to the transversal section H .

Proposition 3.9. *There exist $\epsilon > 0$ and a C^1 map $\rho: B_\epsilon(u(0)) \rightarrow (0, +\infty)$ such that $\rho(u(0)) = p$, $\Sigma(\rho(\varphi), \varphi) \in H$ and $D_1\Sigma(\rho(\varphi), \varphi)1 \notin T_{u(0)}H$ for all $\varphi \in B_\epsilon(u(0))$.*

The C^1 map $\Pi: H \cap B_\epsilon(u(0)) \rightarrow H$ defined by $\Pi(\varphi) := \Sigma(\rho(\varphi), \varphi)$ maps an open subset of H into H and has $u(0)$ as a fixed point. The function Π is called the Poincaré map, or the first recurrence map. It associates to a point φ on a Poincaré section (a hyperplane transversal to the periodic orbit) the point where the orbit through φ first returns to the section.

The stability of the periodic orbit is determined by the eigenvalues of the derivative of Π in $u(0)$, which essentially coincide with Floquet multipliers, as shown in [Theorems 3.10](#) and [3.11](#).

Theorem 3.10. *If $|\lambda| < 1$ for all $\lambda \in \sigma(D\Pi(u(0)))$, then the orbit $u(\mathbb{R})$ is asymptotically stable. If there exists $\lambda \in \sigma(D\Pi(u(0)))$ such that $|\lambda| > 1$, then $u(\mathbb{R})$ is unstable.*

Theorem 3.11. *Excluding 0 and 1, the set of Floquet multipliers of the orbit $u(\mathbb{R})$ coincides with $\sigma(D\Pi(u(0)))$. If 1 is a simple Floquet multiplier, then $1 \notin \sigma(D\Pi(u(0)))$; if it is multiple, then $1 \in \sigma(D\Pi(u(0)))$.*

Corollary 3.12. *Suppose that 1 is a simple Floquet multiplier of the periodic orbit $u(\mathbb{R})$. If all Floquet multipliers except 1 are inside the unit circle, then $u(\mathbb{R})$ is asymptotically stable. If there exists a Floquet multiplier outside the unit circle, then $u(\mathbb{R})$ is unstable.*

In summary, given $u: \mathbb{R} \rightarrow X$ a p -periodic solution of (3.5), the results of Corollary 3.12 are valid if

- (H3.1) the family $\{U_u(t+p, t)\}_{t \in \mathbb{R}}$ determined by (3.6) according to Theorem 3.1 has the spectral isolation property;
- (H3.2) the semiflow Σ is C^1 in a neighborhood of $(p, u(0))$ and $u'(0) = u'(p) \notin T_{u(0)}H$, with $H \subset X$ a hyperplane such that $u(0) \in H$.

3.2 APPLICATION TO RFDEs

Details on the application of the theory of section 3.1 to RFDEs are contained in [40] and are summarized here for the reader's convenience.

In the case of RFDEs, the state space is $Y = C([- \tau, 0], \mathbb{R}^d)$ and the C_0 semigroup $T_0 = \{T_0(t)\}_{t \geq 0}$ is defined, for each $\psi \in Y$ and $\theta \in [- \tau, 0]$, as

$$(T_0(t)\psi)(\theta) := \begin{cases} \psi(0), & \text{if } t + \theta \geq 0, \\ \psi(t + \theta), & \text{if } -\tau \leq t + \theta \leq 0. \end{cases}$$

Then

- $Y^* \cong NBV([0, \tau], \mathbb{R}^d)$, i.e., the space of functions $\zeta: [0, \tau] \rightarrow \mathbb{R}^d$ of bounded variation such that ζ is continuous from the right on $(0, \tau)$ and $\zeta(0) = 0$, with the duality pairing given by the sum of Riemann–Stieltjes integrals

$$\langle \zeta, \psi \rangle = \sum_{i=1}^d \int_0^\tau d\zeta_i(\theta) \psi_i(-\theta);$$

- the adjoint semigroup T_0^* is defined, for each $\zeta \in Y^*$ and $\theta \in [0, \tau]$, as

$$(T_0^*(t)\zeta)(\theta) = \begin{cases} \zeta(t + \theta), & \text{if } 0 < \theta \leq \tau, \\ 0, & \text{if } \theta = 0; \end{cases}$$

- $Y^\odot \cong \mathbb{R}^d \times L^1([0, \tau], \mathbb{R}^d)$;

- the semigroup $T_0^\odot$ is defined, for each $(c, g) \in Y^\odot$, as

$$T_0^\odot(t)(c, g) = \left(c + \int_0^t g(\sigma) d\sigma, g(t + \cdot) \right),$$

with the convention that g is extended to $(h, +\infty)$ by 0;

- $Y^{\odot*} \cong \mathbb{R}^d \times L^\infty([-\tau, 0], \mathbb{R}^d)$, with the duality pairing given by

$$\langle (\alpha, \psi), (c, g) \rangle = \sum_{i=1}^d \left(\alpha_i c_i + \int_0^\tau \psi_i(-\theta) g_i(\theta) d\theta \right);$$

- the semigroup $T_0^{\odot*}$ is defined, for each $(\alpha, \psi) \in Y^{\odot*}$, as

$$(T_0^{\odot*}(t)(\alpha, \psi)) = (\alpha, \psi_t^\alpha),$$

with

$$\psi_t^\alpha(\theta) := \begin{cases} \alpha, & \text{if } t + \theta > 0, \\ \psi(t + \theta), & \text{if } t + \theta \leq 0; \end{cases}$$

- $Y^{\odot\odot} \cong \{(\alpha, \psi) \in \mathbb{R}^d \times L^\infty([-\tau, 0], \mathbb{R}^d) \mid \psi \in C(\alpha)\}$, where $C(\alpha)$ is the closed subspace of $L^\infty([-\tau, 0], \mathbb{R}^d)$ whose elements contain a continuous function with the value α at 0;
- the embedding $j: Y \rightarrow Y^{\odot*}$ is defined as $j(\psi) = (\psi(0), \psi)$, hence $Y^{\odot\odot} = j(Y)$, i.e., Y is sun-reflexive with respect to T_0 .

Consider the autonomous (nonlinear) RFDE

$$y'(t) = g(y_t), \tag{3.7}$$

with $g: Y \rightarrow \mathbb{R}^d$ a C^1 function and $y_t \in Y$ defined as in (2.1). Let $G: Y \rightarrow Y^{\odot*}$ be the function defined as

$$G(\psi) := \sum_{i=1}^d g_i(\psi)(e_i, 0),$$

with $(e_1, \dots, e_d)$ the canonical basis of $\mathbb{R}^d$. Observe that G is C^1 .

The solutions of

$$u(t) = T_0(t)\psi + j^{-1} \left(\int_0^t T_0^{\odot*}(t - \sigma) G(u(\sigma)) d\sigma \right) \tag{3.8}$$

(see (3.5)) are in a one-to-one correspondence with the solutions of

$$\begin{cases} y'(t) = g(y_t), & t \geq 0 \\ y_0 = \bar{\psi} \in Y, \end{cases}$$

given by $u(t) = y_t$. Let $\bar{y}$ be a periodic solution of (3.7) and let $\bar{u}$ be the corresponding solution of (3.8). The family $\{B(t)\}_{t \in \mathbb{R}}$ defined as $B(t) := DG(\bar{u}(t))$ is a strongly continuous family of linear and bounded operators from Y to $Y^{\odot*}$. The solutions of

$$w(t) = T_0(t - s)w(s) + j^{-1} \left(\int_s^t T_0^{\odot*}(t - \sigma) DG(\bar{u}(\sigma)) w(\sigma) d\sigma \right), \tag{3.9}$$

(see (3.6)) are in a one-to-one correspondence with the solutions of the initial value problems associated with the linearization of (3.7) around $\bar{y}$

$$\begin{cases} y'(t) = Dg(\bar{y}_t)y_t, & t \geq s \\ y_s = \psi \in Y. \end{cases}$$

As noted in [40, Exercise XIII.2.3], all iterates $U(t+p, t)^j$ with $jp \geq \tau$ for $U(t+p, t)$ in the evolutionary system associated with (3.9) are compact. Moreover, being g a C^1 function, as noted in [40, section XIV.3], it is possible to choose a hyperplane in Y transversal to the orbit at $u(0)$ such that, provided that $p > \tau$, the semiflow Σ is C^1 on a neighborhood of $(p, u(0))$. Hence hypotheses (H3.1) and (H3.2) are fulfilled and Corollary 3.12 applies, yielding the link between the Floquet multipliers of the linearized problem and the local stability of the periodic orbit.

3.3 APPLICATION TO RES AND COUPLED EQUATIONS

The sun-star calculus has been applied to REs and coupled REs/RFDEs in [34], which contains details on the stability and bifurcation analysis of equilibria. In this section we recapitulate the main results relevant to the application of the theory of section 3.1 and study how to verify the validity of hypotheses (H3.1) and (H3.2). This is only a preliminary study and is the subject of ongoing work by the author and colleagues.

In the case of REs, the state space is $X = L^1([-\tau, 0], \mathbb{R}^d)$ and the C_0 semigroup $T_0 = \{T_0(t)\}_{t \geq 0}$ is defined, for each $\varphi \in X$ and $\theta \in [-\tau, 0]$, as

$$(T_0(t)\varphi)(\theta) := \begin{cases} 0, & \text{if } t + \theta > 0, \\ \varphi(t + \theta), & \text{if } -\tau \leq t + \theta \leq 0. \end{cases}$$

Then

- $X^* \cong L^\infty([0, \tau], \mathbb{R}^d)$, with the duality pairing given by

$$\langle g, \varphi \rangle = \sum_{i=1}^d \int_0^\tau g_i(\theta) \varphi_i(-\theta) d\theta;$$

- the adjoint semigroup T_0^* is defined, for each $g \in X^*$ and $\theta \in [0, \tau]$, as

$$(T_0^*(t)g)(\theta) = \begin{cases} 0, & \text{if } t + \theta > \tau, \\ g(t + \theta), & \text{if } 0 \leq t + \theta \leq \tau; \end{cases}$$

- $X^\odot \cong C_0([0, \tau], \mathbb{R}^d)$, i.e., the space of continuous functions vanishing at τ (observe the half-open interval domain);
- $X^{\odot*} \cong NBV((-\tau, 0], \mathbb{R}^d)$ (observe again the half-open interval domain, which implies that there is no jump at $-\tau$), with the duality pairing given by the sum of Riemann–Stieltjes integrals

$$\langle f, g \rangle = \sum_{i=1}^d \int_{-\tau}^0 f_i(d\theta) g_i(-\theta);$$

- the semigroup $T_0^{\odot*}$ is defined, for each $f \in X^{\odot*}$ and $\theta \in (-\tau, 0]$, as

$$(T_0^{\odot*}(t)f)(\theta) := \begin{cases} 0, & \text{if } t + \theta > 0, \\ f(t + \theta), & \text{if } -\tau < t + \theta \leq 0; \end{cases}$$

- $X^{\odot\odot} \cong \{f \in X^{\odot*} \mid f \in AC((-\tau, 0], \mathbb{R}^d)\}$, where $AC((-\tau, 0], \mathbb{R}^d)$ is the space of absolutely continuous functions on $(-\tau, 0]$;
- the embedding $j: X \rightarrow X^{\odot*}$ is defined, for $\varphi \in X$ and $\theta \in (-\tau, 0]$, as

$$j(\varphi)(\theta) = - \int_{\theta}^0 \varphi(\sigma) d\sigma,$$

(which implies that $j(\varphi)' = \varphi$), hence $X^{\odot\odot} = j(X)$, i.e., X is sun-reflexive with respect to T_0 [34, Proposition 3.3].

Consider the autonomous (nonlinear) RE

$$x(t) = f(x_t), \tag{3.10}$$

with $f: X \rightarrow \mathbb{R}^d$ a function and $x_t \in X$ defined as in (2.1). Let $F: X \rightarrow X^{\odot*}$ be the function defined as

$$F(\psi) := \sum_{i=1}^d f_i(\psi) H_i,$$

where, for each $i \in \{1, \dots, d\}$ and $\theta \in (-\tau, 0]$, H_i is defined as

$$H_i(\theta) := \begin{cases} e_i, & \text{if } -\tau < \theta < 0, \\ 0, & \text{if } \theta = 0, \end{cases}$$

with $(e_1, \dots, e_d)$ the canonical basis of $\mathbb{R}^d$. Observe that F is C^1 (in the Fréchet sense) if and only if f is C^1 .

The next theorem shows that also in the case of REs there is a one-to-one correspondence between solutions of the RE and of the associated abstract equation.

Theorem 3.13 ([34, Theorem 3.7]). *Let $\varphi \in X$. If $x \in L_{\text{loc}}^1([-\tau, +\infty), \mathbb{R}^d)$ satisfies*

$$\begin{cases} x(t) = f(x_t), & t > 0, \\ x_0 = \varphi, \end{cases} \tag{3.11}$$

then the function $u: [0, +\infty) \rightarrow X$ defined by $u(t) := x_t$ is continuous and satisfies

$$u(t) = T_0(t)\varphi + j^{-1}\left(\int_0^t T_0^{\odot*}(t - \sigma)F(u(\sigma)) d\sigma\right). \tag{3.12}$$

Vice versa, if there is a continuous function $u: [0, +\infty) \rightarrow X$ that satisfies (3.12), then the function $x: [0 - \tau, +\infty) \rightarrow \mathbb{R}^d$ defined as

$$x(t) := \begin{cases} u(t)(0), & \text{if } t \geq 0, \\ \varphi(t), & \text{if } -h \leq t < 0, \end{cases}$$

is in $L_{\text{loc}}^1([-\tau, +\infty), \mathbb{R}^d)$ and satisfies (3.11).

Let $\bar{x}$ be a periodic solution of (3.11) and let $\bar{u}$ be the corresponding solution of (3.12). Define the family $\{B(t)\}_{t \in \mathbb{R}}$ as $B(t) := DF(\bar{u}(t))$ and observe that each $B(t): X \rightarrow X^{\odot*}$ is a linear and bounded functional by definition of Fréchet derivative.

The family $\{B(t)\}_{t \in \mathbb{R}}$ is strongly continuous, i.e., for every $\varphi \in X$ the function $t \mapsto B(t)\varphi$ is continuous from $\mathbb{R}$ to $X^{\odot*}$. Indeed, by definition of being C^1 in the sense of Fréchet, $\varphi \rightarrow DF(\varphi)$ is continuous in the operator norm $\|\cdot\|_{X^{\odot*} \leftarrow X}$. Thus, by the continuity of $\bar{u}(t)$ the function $t \mapsto B(t)$ is continuous in the operator norm. Being $B(t)$ bounded,

$$\|B(t_1)\varphi - B(t_0)\varphi\|_{X^{\odot*}} \leq \|B(t_1) - B(t_0)\|_{X^{\odot*} \leftarrow X} \|\varphi\|_X,$$

hence the continuity of $t \mapsto B(t)\varphi$.

Similarly to the nonlinear case of Theorem 3.13, the solutions of

$$w(t) = T_0(t-s)w(s) + j^{-1} \left(\int_s^t T_0^{\odot*}(t-\sigma) DF(\bar{u}(\sigma)) w(\sigma) d\sigma \right) \quad (3.13)$$

(see (3.6)) are in a one-to-one correspondence with the solutions of the initial value problems associated with the linearization of (3.10) around $\bar{x}$

$$\begin{cases} x(t) = Df(\bar{x}_t)x_t, & t \geq s \\ x_s = \varphi \in X. \end{cases} \quad (3.14)$$

The linearized RE can be written as

$$x(t) = \int_{-\tau}^0 K(t, \theta) x(t + \theta) d\theta,$$

or, equivalently,

$$x(t) = \int_{t-\tau}^t K(t, \sigma - t) x(\sigma) d\sigma,$$

for $K: \mathbb{R} \times [-\tau, 0] \rightarrow \mathbb{R}^{d \times d}$ a measurable function, periodic in t , thanks to the Riesz representation theorem for L^1 (Theorem 2.30).

VERIFYING HYPOTHESIS (H3.1). In order to prove the validity of hypothesis (H3.1) (under suitable hypotheses on the kernel of the linearized equation), i.e., that the family of period evolution operators $\{U_{\bar{u}}(t+p, t)\}_{t \in \mathbb{R}}$ determined by (3.13) according to Theorem 3.1 has the spectral isolation property, we resort to the Kolmogorov–M. Riesz–Fréchet theorem (Theorem 2.31).

Let $\tilde{p} := \max\{p, \tau\}$ and assume that the interval $[0, \tilde{p}]$ can be partitioned into finitely many subintervals $J_1, \dots, J_n$ such that, for any $s \in \mathbb{R}$,

$$\operatorname{ess\,sup}_{\sigma \in J_i} \int_{J_i} |K(s+t, \sigma-t)| dt < 1, \quad i \in \{1, \dots, n\},$$

with K prolonged by 0 where it is not defined. For simplicity of notation, let $s = 0$.

For $t \in [0, \tilde{p}]$, define

$$f(t) := \begin{cases} \int_{t-\tau}^0 K(t, \sigma-t) \varphi(\sigma) d\sigma, & \text{if } t \leq \tau, \\ 0, & \text{otherwise.} \end{cases}$$

We can write

$$x(t) = \int_0^t K(t, \sigma - t)x(\sigma) d\sigma + f(t).$$

For $(t, \sigma) \in [0, \tilde{p}]^2$, define

$$\tilde{K}(t, \sigma) := \begin{cases} K(t, \sigma - t), & \text{if } \sigma \in [t - \tau, t], \\ 0, & \text{otherwise,} \end{cases}$$

and write

$$x(t) = \int_0^t \tilde{K}(t, \sigma)x(\sigma) d\sigma + f(t).$$

Let $\tilde{R}$ be the resolvent of $\tilde{K}$ given by [Theorem 2.27](#). By [Theorem 2.26](#), the solution $x(t)$ is given by

$$x(t) = \int_0^t \tilde{R}(t, \sigma)f(\sigma) d\sigma + f(t)$$

for $t \in [0, \tilde{p}]$, which can be rewritten as

$$x(t) = \int_0^t R(t, \sigma - t)f(\sigma) d\sigma + f(t) \quad (3.15)$$

by defining, for $(t, \theta) \in [0, \tilde{p}] \times [-\tilde{p}, 0]$,

$$R(t, \theta) := \begin{cases} \tilde{R}(t, t + \theta), & \text{if } \theta \in [-t, 0], \\ 0, & \text{otherwise.} \end{cases}$$

Consider the operator $U := U_{\tilde{a}}(p, 0)$. By the correspondence between solutions of [\(3.13\)](#) and [\(3.14\)](#) and by [Theorem 3.1](#), for $\varphi \in X$

$$U\varphi = x_p(\cdot; 0, \varphi),$$

where $x(\cdot; 0, \varphi)$ is the solution of [\(3.14\)](#) with initial value φ at time 0.

The objective is to prove that U has the spectral isolation property. This is ensured if U is compact, hence the aim is to prove that the image of the unit ball in X under U is relatively compact in X . Let

$$\Phi := \{\varphi \in X \mid \|\varphi\|_X \leq 1\}$$

and let $\overline{U\Phi}$ be the set of prolongations of functions in $U\Phi$ to $\mathbb{R}$ by 0. Observe that $\overline{U\Phi}$ is bounded, since U and Φ are bounded. By the Kolmogorov–M. Riesz–Fréchet theorem ([Theorem 2.31](#)), we need to prove that

$$\lim_{\eta \rightarrow 0} \|\tau_\eta \psi - \psi\|_{L^1(\mathbb{R}, \mathbb{R}^d)} = 0$$

uniformly in $\psi \in \overline{U\Phi}$, i.e., for each $\epsilon > 0$ there exists $\delta > 0$ such that $\|\tau_\eta \psi - \psi\|_{L^1(\mathbb{R}, \mathbb{R}^d)} < \epsilon$ for all $\psi \in \overline{U\Phi}$ and all $\eta \in \mathbb{R}$ such that $|\eta| < \delta$, where τ_η denotes the translation by η defined by $(\tau_\eta \psi)(t) := \psi(t + \eta)$.

Let $\psi \in \overline{U\Phi}$. Then there exists $\varphi \in \Phi$ such that for $t \in \mathbb{R}$

$$\psi(t) = \begin{cases} (U\varphi)(t), & \text{if } t \in [-\tau, 0], \\ 0, & \text{otherwise.} \end{cases}$$

Observe that $\|\psi\|_{L^1(\mathbb{R}, \mathbb{R}^d)} = \|U\varphi\|_X$ and for $\theta \in [-\tau, 0]$

$$(U\varphi)(\theta) = \int_0^{\theta+p} R(\theta+p, \sigma-\theta-p) f(\sigma) d\sigma + f(\theta+p),$$

thanks to (3.15).

Suppose $p \geq \tau$ and $\eta > 0$. Then

$$\begin{aligned} \|\tau_\eta \psi - \psi\|_{L^1(\mathbb{R}, \mathbb{R}^d)} &= \int_{-\infty}^{+\infty} |\psi(\theta+\eta) - \psi(\theta)| d\theta \\ &= \int_{-\tau}^{-\eta} |(U\varphi)(\theta+\eta) - (U\varphi)(\theta)| d\theta \\ &\quad + \int_{-\tau-\eta}^{-\tau} |(U\varphi)(\theta+\eta)| d\theta + \int_{-\eta}^0 |(U\varphi)(\theta)| d\theta \\ &= \int_{-\tau}^{-\eta} |(U\varphi)(\theta+\eta) - (U\varphi)(\theta)| d\theta \\ &\quad + \int_{-\tau}^{-\tau+\eta} |(U\varphi)(\theta)| d\theta + \int_{-\eta}^0 |(U\varphi)(\theta)| d\theta. \end{aligned}$$

The last two terms converge to 0 as $\eta \rightarrow 0$. As for the first term,

$$\begin{aligned} &\int_{-\tau}^{-\eta} |(U\varphi)(\theta+\eta) - (U\varphi)(\theta)| d\theta \\ &\leq \int_{-\tau}^{-\eta} |f(\theta+\eta+p) - f(\theta+p)| d\theta \\ &\quad + \int_{-\tau}^{-\eta} \int_{\theta+p}^{\theta+\eta+p} |\tilde{R}(\theta+\eta+p, \sigma) f(\sigma)| d\sigma d\theta \\ &\quad + \int_{-\tau}^{-\eta} \int_0^{\theta+p} |[\tilde{R}(\theta+\eta+p, \sigma) - \tilde{R}(\theta+p, \sigma)] f(\sigma)| d\sigma d\theta. \end{aligned}$$

By the continuity of translation in L^1 the first term converges to 0 as $\eta \rightarrow 0$. For the second term,

$$\begin{aligned} &\int_{-\tau}^{-\eta} \int_{\theta+p}^{\theta+\eta+p} |\tilde{R}(\theta+\eta+p, \sigma) f(\sigma)| d\sigma d\theta \\ &= \int_{-\tau+p}^p \int_{\max\{-\tau, \sigma-p\}}^{\min\{-\eta, \sigma-p\}} |\tilde{R}(\theta+\eta+p, \sigma)| d\theta |f(\sigma)| d\sigma. \end{aligned}$$

Observe that as $\eta \rightarrow 0$, the integration interval of the inner integral vanishes. By Lebesgue's dominated convergence theorem (Theorem 2.29), this term converges to 0. The third term can be rewritten as

$$\begin{aligned} &\int_{-\tau}^{-\eta} \int_0^{\theta+p} |[\tilde{R}(\theta+\eta+p, \sigma) - \tilde{R}(\theta+p, \sigma)] f(\sigma)| d\sigma d\theta \\ &= \int_0^{p-\eta} \int_{\max\{-\tau, \sigma-p\}}^{-\eta} |[\tilde{R}(\theta+\eta+p, \sigma) - \tilde{R}(\theta+p, \sigma)]| d\theta |f(\sigma)| d\sigma \\ &\leq \int_0^p \int_{\max\{-\tau, \sigma-p\}}^0 |[\tilde{R}(\theta+\eta+p, \sigma) - \tilde{R}(\theta+p, \sigma)]| d\theta |f(\sigma)| d\sigma, \end{aligned}$$

which, again by the continuity of translation in L^1 and Lebesgue's dominated convergence theorem, converges to 0 as $\eta \rightarrow 0$.

If $p < \tau$, the integrals in $\|\tau_\eta \psi - \psi\|_{L^1(\mathbb{R}, \mathbb{R}^d)}$ need to be split differently, since part of them concerns the initial value φ : those terms converge to 0 for the

continuity of translation in L^1 . The remaining terms are tackled similarly as before and similar results can be obtained for $\eta < 0$ and for other operators of the family, hence $\|\tau_\eta \psi - \psi\|_{L^1(\mathbb{R}, \mathbb{R}^d)}$ converges to 0 as $\eta \rightarrow 0$.

As anticipated, this is only a preliminary and incomplete study. In order to complete the proof, we would need to show that the convergence is uniform in $\psi \in \overline{U\Phi}$. By rewriting f in terms of its definition, the relevant limits should depend only on the kernel and the resolvent and not on the choice of the function φ .

VERIFYING HYPOTHESIS (H3.2). In order to prove the desired link between the Floquet multipliers of the linearized problem and the local stability of the periodic orbit, it remains to prove the validity of [hypothesis \(H3.2\)](#). This requires the semiflow Σ to be Fréchet differentiable on a neighborhood of $(p, u(0))$ and the function $(t, \varphi) \mapsto D\Sigma(t, \varphi)$ to be continuous in the operator norm from the space $\mathbb{R} \times X$ to the space of linear and bounded operators $\mathbb{R} \times X \rightarrow X$. It implies also the differentiability of $u: \mathbb{R} \rightarrow X$. This requirement has a twofold importance: first, the differentiability of u implies that 1 is a Floquet multiplier (see [Proposition 3.8](#)); second, the conditions on the semiflow Σ ensure the existence of a hyperplane transversal to the solution, allowing to define the Poincaré map. For both purposes, it seems reasonable that with sufficient regularity of the integration kernel the hypothesis can be validated. Recall that the state space X is a space of L^1 functions, hence the continuity and differentiability properties involve convergence in the norms of L^1 and related operator norms: this suggests that requirements less strict than in the RFDE case may be sufficient.

A further hint in the direction of the validity of the results in this chapter for REs can be seen in the numerical tests of [chapter 8](#): indeed, in the examples of equations linearized around periodic solutions, the Floquet multiplier 1 is always present.

In [\[34, section 4\]](#) it is shown that extending sun-star calculus and the results on stability for equilibria from RFDEs and REs to coupled equations is rather straightforward. It is reasonable to speculate that this is the case also for the results on Floquet theory and Poincaré maps.

As already anticipated, the results presented in this section are only preliminary. The completion of the extension of Floquet theory to REs and coupled equations is the subject of ongoing research by the author and colleagues (see also [chapter 9](#)).

4

RETARDED FUNCTIONAL DIFFERENTIAL EQUATIONS

Most of the material covered in this chapter is taken from [15] and [16, chapter 6]. It seemed sensible to present it again here for the reader's convenience, in order to better underline the differences between the method for RFDEs and its versions for REs and coupled REs/RFDEs in the next chapters.

There is however one notable difference with respect to the original exposition. In the cited works in order to prove the convergence of the approximated eigenvalues to the real ones, the relevant operators, naturally posed on the state space of continuous functions, were restricted to Lipschitz continuous functions. In this new exposition, instead, the operators are restricted to absolutely continuous functions. The advantage of this change is the possibility of requiring less stringent hypotheses on the coefficients of the RFDE (compare [hypotheses \(H4.3\)](#) and [\(H4.4\)](#) to conditions (C1) and (C3) in [15, Theorem 3.3 and Proposition 4.5] and [16, Theorem 6.1 and Proposition 6.2]).

This change has been possible thanks to [Theorem 2.17](#), a result by Krylov published only in Russian in [65], which gives the required result on the convergence of Lagrange interpolation. When the original authors of [15] developed the method, they were not yet aware of this result, which has been brought to their attention only recently for the work in [13]. Thus, they employed classical results in interpolation theory (see, e.g., [83, Corollary 1.4.2, Theorems 4.1 and 4.5]) to obtain the convergence of Lagrange interpolation on subspaces of Lipschitz continuous functions. It seemed reasonable in this new exposition to adapt the proofs to the absolutely continuous case, since to the best of the author's knowledge it is the least degree of regularity to require that allows to complete the proof of convergence for RFDEs.

4.1 EVOLUTION OPERATORS FOR LINEAR RFDEs

Let $d \in \mathbb{N}$ and $\tau \in \mathbb{R}$, both positive, and consider the function space

$$Y := C([- \tau, 0], \mathbb{R}^d)$$

equipped with the usual uniform norm

$$\|\psi\|_Y := \max_{\theta \in [-\tau, 0]} |\psi(\theta)|. \quad (4.1)$$

A linear RFDE with finite delay is a relation of the form

$$y'(t) = L(t)y_t, \quad t \in \mathbb{R}, \quad (4.2)$$

where y' denotes the right-hand derivative of y , y_t is defined as in [\(2.1\)](#) and $\mathbb{R} \times Y \ni (t, \psi) \mapsto L(t)\psi \in \mathbb{R}^d$ is a continuous function, linear in the second

argument. This condition implies that $L(t): Y \rightarrow \mathbb{R}^d$ is a linear bounded functional for all $t \in \mathbb{R}$ and $L(\cdot)\psi: \mathbb{R} \rightarrow \mathbb{R}^d$ is a continuous function for all $\psi \in Y$. The numbers d and τ are, respectively, the dimension of the equation and the maximum delay, while Y is the state space and $y_t \in Y$ is the state at time t .

For $s \in \mathbb{R}$ and $\psi \in Y$, the Cauchy problem for (4.2) is defined as

$$\begin{cases} y'(t) = L(t)y_t, & t \geq s, \\ y_s = \psi. \end{cases} \quad (4.3)$$

A function y is a solution of (4.3) on $[s - \tau, s + t_f)$ if there exists $t_f > 0$ such that $y \in C([s - \tau, s + t_f), \mathbb{R}^d)$, $y_s = \psi$, and $y(t)$ satisfies (4.2) for each $t \in [s, s + t_f)$. The final time t_f may be $+\infty$. To emphasize the dependence of solutions on both the initial time s and the initial function ψ , a solution $y(\cdot)$ of (4.3) is sometimes denoted as $y(\cdot; s, \psi)$.

Thanks to [56, Lemma 2.1.1], the continuity of $(t, \psi) \mapsto L(t)\psi$ ensures that solving (4.3) is equivalent to solving

$$\begin{cases} y(t) = \psi(0) + \int_s^t L(\sigma)y_\sigma d\sigma, & t \geq s, \\ y_s = \psi. \end{cases}$$

The following is a classical result on the existence and uniqueness of solutions for RFDEs (see, e.g., [56, Theorems 2.2.1, 2.2.2 and 2.2.3] and [90, Theorem 3.7 and Remark 3.8]). A function $f: \mathbb{R} \times Y \rightarrow \mathbb{R}^d$ is Lipschitz continuous with respect to Y on $D \subset \mathbb{R} \times Y$ if there exists a constant $\text{Lip}(f; D)$ such that for each $(t, \psi_1), (t, \psi_2) \in D$

$$|f(t, \psi_1) - f(t, \psi_2)| \leq \text{Lip}(f; D) \|\psi_1 - \psi_2\|_Y;$$

it is globally Lipschitz continuous with respect to Y if it is Lipschitz continuous with respect to Y on $\mathbb{R} \times Y$.

Theorem 4.1. *Let $f: \mathbb{R} \times Y \rightarrow \mathbb{R}^d$ be a continuous function, Lipschitz continuous with respect to Y in each compact set in $\mathbb{R} \times Y$. If $(s, \psi) \in \mathbb{R} \times Y$, then there exist $t_f > 0$ and a unique solution on $[s - \tau, s + t_f)$ of*

$$\begin{cases} y'(t) = f(t, y_t), & t \geq s, \\ y_s = \psi. \end{cases}$$

The solution y depends continuously on the initial data s, ψ and f , in the sense that if $\{(s^{(k)}, \psi^{(k)}, f^{(k)})\}_{k \in \mathbb{N}}$ is a sequence such that

- $s^{(k)} \rightarrow s$,
- $\|\psi^{(k)} - \psi\|_Y \rightarrow 0$,
- $\sup |f^{(k)}(t, \psi) - f(t, \psi)| \rightarrow 0$ on a subset of $\mathbb{R} \times Y$ containing $\{(t, y_t) \mid t \in [s, s + t_0]\}$ for some $0 < t_0 < t_f$ on which f is bounded,

*then there exists $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$ the Cauchy problem defined by $(s^{(k)}, \psi^{(k)}, f^{(k)})$ admits a unique solution $y^{(k)}$ on $[s^{(k)} - \tau, s + t_0]$ and $y^{(k)} \rightarrow y$ uniformly on $[s - \tau, s + t_0]$.**

* Since for some k the solution $y^{(k)}$ may not be defined on $[s - \tau, s + t_0]$, by $y^{(k)} \rightarrow y$ uniformly on $[s - \tau, s + t_0]$ we mean, according to [56, Theorem 2.2.2], that for each $\epsilon > 0$ there exists $k_1(\epsilon)$ such that for $k \geq k_1(\epsilon)$ the solution $y^{(k)}$ is defined on $[s - \tau + \epsilon, s + t_0]$ and $y^{(k)} \rightarrow y$ uniformly on $[s - \tau + \epsilon, s + t_0]$.

Moreover, if f is globally Lipschitz continuous with respect to Y , then the solution exists uniquely on $[s - \tau, +\infty)$.

By [Theorem 4.1](#), if $t \mapsto \|L(t)\|_{\mathbb{R}^d \leftarrow Y}$ is bounded (as is the case, e.g., if $L(t)$ is periodic), for each $s \in \mathbb{R}$ and $\psi \in Y$ the Cauchy problem [\(4.3\)](#) admits a unique solution on $[s - \tau, +\infty)$. This allows us to define the family $\{T(t, s)\}_{(t, s) \in \Delta}$ of evolution operators

$$T(t, s): Y \rightarrow Y, \quad T(t, s)\psi := y_t(\cdot; s, \psi), \quad (4.4)$$

where

$$\Delta := \{(t, s) \in \mathbb{R}^2 \mid -\infty \leq s \leq t \leq +\infty\}.$$

The following proposition is a consequence of [Theorem 3.1](#) and of the correspondence between solutions of the linear initial value problem and the relevant abstract equation (see [section 3.2](#)), but it can also be proved directly.

Proposition 4.2. *If $t \mapsto \|L(t)\|_{\mathbb{R}^d \leftarrow Y}$ is bounded, the family of evolution operators $\{T(t, s)\}_{(t, s) \in \Delta}$ defined in [\(4.4\)](#) is a strongly continuous evolutionary system.*

Proof. Recall the definition of evolutionary system from [subsection 3.1.2](#). For each $(t, s) \in \Delta$ the operator $T(t, s)$ is linear since the RFDE [\(4.2\)](#) is linear. The property [\(3.1\)](#) is obvious from the definition, while the property [\(3.2\)](#) holds by uniqueness of solutions. For $\psi \in Y$ and $\theta \in [-\tau, 0]$, observe that

$$(T(t, s)\psi)(\theta) = \begin{cases} \psi(0) + \int_s^{t+\theta} L(\sigma)T(\sigma, s)\psi \, d\sigma, & t + \theta \geq s, \\ \psi(t + \theta - s), & t + \theta < s. \end{cases}$$

Hence,

$$\|T(t, s)\psi\|_Y \leq \|\psi\|_Y + \int_s^t \bar{L} \|T(\sigma, s)\psi\|_Y \, d\sigma,$$

where $\bar{L} := \sup_{t \in \mathbb{R}} \|L(t)\|_{\mathbb{R}^d \leftarrow Y}$. Observe that thanks to the continuity of the solution, for each $s \in \mathbb{R}$ and $\psi \in Y$ the function $\|T(\cdot, s)\psi\|_Y$ is continuous. By Grönwall's inequality ([Theorem 2.32](#)), for $(t, s) \in \Delta$,

$$\|T(t, s)\psi\|_Y \leq \|\psi\|_Y \exp\left(\int_s^t \bar{L} \, d\sigma\right) = \|\psi\|_Y e^{\bar{L}(t-s)},$$

from which follows the boundedness of $T(t, s)$. Finally, consider a sequence $\{(t_n, s_n)\}_{n \in \mathbb{N}} \subset \Delta$ such that $(t_n, s_n) \rightarrow (\bar{t}, \bar{s})$ and observe that given $\psi \in Y$

$$\begin{aligned} \|T(t_n, s_n)\psi - T(\bar{t}, \bar{s})\psi\|_Y &= \|y_{t_n}(\cdot; s_n, \psi) - y_{\bar{t}}(\cdot; \bar{s}, \psi)\|_Y \\ &\leq \|y_{t_n}(\cdot; s_n, \psi) - y_{t_n}(\cdot; \bar{s}, \psi)\|_Y \\ &\quad + \|y_{t_n}(\cdot; \bar{s}, \psi) - y_{\bar{t}}(\cdot; \bar{s}, \psi)\|_Y. \end{aligned}$$

The first term converges to 0 thanks to the continuous dependence on initial time, while the second converges to 0 thanks to the continuity of the solution, which implies that it is uniformly continuous on compact sets. $\square$

Let $s \in \mathbb{R}$ and $h \geq 0$ and consider the evolution operator

$$T := T(s + h, s).$$

The aim of this chapter is to approximate the spectrum of T by computing with standard techniques the eigenvalues of a finite-dimensional approximation of T obtained via pseudospectral collocation, as described in [section 4.3](#).

Recall from [section 1.2](#) that this allows to study the stability of equilibria (by studying the eigenvalues of $T(h, 0)$ of the linearized problem for any $h > 0$) and periodic solutions (by studying the spectrum of $T(\Omega, 0)$ of the Ω -periodic linearized problem), and that this discretization technique can be applied to the approximation of Lyapunov exponents for generic nonautonomous linear equations, as mentioned also in [chapter 9](#).

4.2 REFORMULATION OF T

Before we proceed to the discretization of T , we aim at reformulating it by means of two other suitably defined operators. This reformulation is convenient both for the proof of convergence and for the implementation, since it allows to separate different aspects of the equation, i.e., the ones related to the specific equation considered and the ones related only to the type of equation.

Define the function spaces

$$Y^+ := C([0, h], \mathbb{R}^d), \quad Y^\pm := C([- \tau, h], \mathbb{R}^d),$$

equipped with the corresponding uniform norms denoted, respectively, by $\|\cdot\|_{Y^+}$ and $\|\cdot\|_{Y^\pm}$.

Define the operator $V: Y \times Y^+ \rightarrow Y^\pm$ as

$$V(\psi, z)(t) := \begin{cases} \psi(0) + \int_0^t z(\sigma) d\sigma, & t \in (0, h], \\ \psi(t), & t \in [-\tau, 0]. \end{cases} \quad (4.5)$$

Let $V^-: Y \rightarrow Y^\pm$ and $V^+: Y^+ \rightarrow Y^\pm$ be given, respectively, by $V^-\psi := V(\psi, 0_{Y^+})$ and $V^+z := V(0_Y, z)$. Observe that

$$V(\psi, z) = V^-\psi + V^+z. \quad (4.6)$$

The operator V captures the rule to construct a solution of [\(4.3\)](#), given the initial value ψ and the data obtained from the equation, namely the derivative z of the solution.

Define the operator $\mathcal{F}_s: Y^\pm \rightarrow Y^+$ as

$$\mathcal{F}_s v(t) := L(s + t)v_t, \quad t \in [0, h]. \quad (4.7)$$

The operator $\mathcal{F}_s$ applies the right-hand side functional to its argument after a time shift, in order to operate between spaces of functions defined always on the same respective time intervals.

The evolution operator T can be reformulated as

$$T\psi = V(\psi, z^*)_h, \quad (4.8)$$

where $z^* \in Y^+$ is the solution of the fixed point equation

$$z = \mathcal{F}_s V(\psi, z), \quad (4.9)$$

which exists uniquely and bounded thanks to [Corollary 4.3](#) below. Recall that in (4.8) the subscript h is used according to [Definition 2.1](#), hence

$$V(\psi, z^*)_h(\theta) = V(\psi, z^*)(h + \theta)$$

for $\theta \in [-\tau, 0]$.

The next result is a corollary of [Theorem 4.1](#) which will be useful later on.

Corollary 4.3. *If $t \mapsto \|L(t)\|_{\mathbb{R}^d \leftarrow Y}$ is bounded, then the operator $I_{Y^+} - \mathcal{F}_s V^+$ is invertible with bounded inverse and (4.9) admits a unique solution in Y^+ .*

Proof. Given $g \in Y^+$, the equation $(I_{Y^+} - \mathcal{F}_s V^+)z = g$ has a unique solution $z \in Y^+$ if and only if the initial value problem

$$\begin{cases} y'(t) = L(s+t)y_t + g(t), & t \in [0, h], \\ y_0 = 0 \in Y, \end{cases}$$

has a unique solution y in $Y^\pm$, with z and y' coinciding on $[0, h]$. This follows from [Theorem 4.1](#). So $I_{Y^+} - \mathcal{F}_s V^+$ is invertible and bounded and the bounded inverse theorem ([Theorem 2.19](#)) completes the proof. $\square$

4.3 DISCRETIZATION

The aim is now to discretize T . We apply a pseudospectral technique to the reformulation (4.8) and (4.9). In this section we describe the meshes of interpolation nodes in the relevant time intervals and the discretization of function spaces and of T by collocation based on Lagrange interpolating polynomials.

Let M and N be positive integers, referred to as discretization indices.

4.3.1 Partition of time intervals

If $h \geq \tau$, let $\Omega_M := \{\theta_{M,0}, \dots, \theta_{M,M}\}$ be a partition of $[-\tau, 0]$ with

$$-\tau = \theta_{M,M} < \dots < \theta_{M,0} = 0.$$

If $h < \tau$, instead, let Q be the minimum positive integer q such that $qh \geq \tau$. Note that $Q > 1$. Let $\theta^{(q)} := -qh$ for $q \in \{0, \dots, Q-1\}$ and $\theta^{(Q)} := -\tau$. For $q \in \{1, \dots, Q\}$, let $\Omega_M^{(q)} := \{\theta_{M,0}^{(q)}, \dots, \theta_{M,M}^{(q)}\}$ be a partition of $[\theta^{(q)}, \theta^{(q-1)}]$ with

$$\begin{aligned} \theta^{(1)} &= \theta_{M,M}^{(1)} < \dots < \theta_{M,0}^{(1)} = \theta^{(0)} = 0, \\ \theta^{(q)} &= \theta_{M,M}^{(q)} < \dots < \theta_{M,0}^{(q)} = \theta^{(q-1)}, & q \in \{2, \dots, Q-1\}, \\ -\tau &= \theta^{(Q)} = \theta_{M,M}^{(Q)} < \dots < \theta_{M,0}^{(Q)} = \theta^{(Q-1)}. \end{aligned}$$

Define also the partition $\Omega_M := \Omega_M^{(1)} \cup \dots \cup \Omega_M^{(Q)}$ of $[-\tau, 0]$. Note in particular that for $q \in \{1, \dots, Q-1\}$

$$\theta_{M,M}^{(q)} = -qh = \theta_{M,0}^{(q+1)}. \quad (4.10)$$

In principle, one can use more general meshes in $[-\tau, 0]$, e.g., not including the endpoints or using different families of nodes in the piecewise case. The forthcoming results can be generalized straightforwardly, but we avoid this choice in favor of a lighter notation and to reduce technicalities.

Finally, let $\Omega_N^+ := \{t_{N,1}, \dots, t_{N,N}\}$ be a partition of $[0, h]$ with

$$0 \leq t_{N,1} < \dots < t_{N,N} \leq h.$$

4.3.2 Discretization of function spaces

If $h \geq \tau$, the discretization of Y of index M is $Y_M := \mathbb{R}^{d(M+1)}$. An element $\Psi \in Y_M$ is written as $\Psi = (\Psi_0, \dots, \Psi_M)$, where $\Psi_m \in \mathbb{R}^d$ for $m \in \{0, \dots, M\}$. The restriction operator $R_M: Y \rightarrow Y_M$ is given by

$$R_M \psi := (\psi(\theta_{M,0}), \dots, \psi(\theta_{M,M})).$$

The prolongation operator $P_M: Y_M \rightarrow Y$ is the discrete Lagrange interpolation operator

$$P_M \Psi(\theta) := \sum_{m=0}^M \ell_{M,m}(\theta) \Psi_m, \quad \theta \in [-\tau, 0],$$

where $\ell_{M,0}, \dots, \ell_{M,M}$ are the Lagrange coefficients relevant to the nodes of Ω_M (recall [subsection 2.3.2](#) for the definition of Lagrange coefficients). Observe that

$$R_M P_M = I_{Y_M}, \quad P_M R_M = \mathcal{L}_M, \quad (4.11)$$

where $\mathcal{L}_M: Y \rightarrow Y$ is the Lagrange interpolation operator that associates to a function $\psi \in Y$ the M -degree $\mathbb{R}^d$ -valued polynomial $\mathcal{L}_M \psi$ such that

$$\mathcal{L}_M \psi(\theta_{M,m}) = \psi(\theta_{M,m})$$

for $m \in \{0, \dots, M\}$.

If $h < \tau$, proceed similarly but in a piecewise fashion. The discretization of Y of index M is $Y_M := \mathbb{R}^{d(QM+1)}$. An element $\Psi \in Y_M$ is written as

$$\Psi = (\Psi_0^{(1)}, \dots, \Psi_{M-1}^{(1)}, \dots, \Psi_0^{(Q)}, \dots, \Psi_{M-1}^{(Q)}, \Psi_M^{(Q)}), \quad (4.12)$$

where $\Psi_m^{(q)} \in \mathbb{R}^d$ for $q \in \{1, \dots, Q\}$ and $m \in \{0, \dots, M-1\}$ and $\Psi_M^{(Q)} \in \mathbb{R}^d$. In view of [\(4.10\)](#), let also $\Psi_M^{(q)} := \Psi_0^{(q+1)}$ for $q \in \{1, \dots, Q-1\}$. The restriction operator $R_M: Y \rightarrow Y_M$ is given by

$$R_M \psi := (\psi(\theta_{M,0}^{(1)}), \dots, \psi(\theta_{M,M-1}^{(1)}), \dots, \psi(\theta_{M,0}^{(Q)}), \dots, \psi(\theta_{M,M-1}^{(Q)}), \psi(\theta_{M,M}^{(Q)})).$$

The prolongation operator $P_M: Y_M \rightarrow Y$ is the discrete piecewise Lagrange interpolation operator

$$P_M \Psi(\theta) := \sum_{m=0}^M \ell_{M,m}^{(q)}(\theta) \Psi_m^{(q)}, \quad \theta \in [\theta^{(q)}, \theta^{(q+1)}], \quad q \in \{1, \dots, Q\},$$

where $\ell_{M,0}^{(q)}, \dots, \ell_{M,M}^{(q)}$ are the Lagrange coefficients relevant to the nodes of $\Omega_M^{(q)}$ for $q \in \{1, \dots, Q\}$. Observe that the equalities [\(4.11\)](#) hold again, with

$\mathcal{L}_M: Y \rightarrow Y$ the piecewise Lagrange interpolation operator that associates to a function $\psi \in Y$ the piecewise polynomial $\mathcal{L}_M\psi$ such that $\mathcal{L}_M\psi|_{[\theta^{(q)}, \theta^{(q-1)})}$ is the M -degree $\mathbb{R}^d$ -valued polynomial with values $\psi(\theta_{M,m}^{(q)})$ at the nodes $\theta_{M,m}^{(q)}$ for $q \in \{1, \dots, Q\}$ and $m = 0, \dots, M$. Notice that to avoid a cumbersome notation the same symbols for Y_M , R_M , P_M and $\mathcal{L}_M$ are used.

Finally, the discretization of Y^+ of index N is $Y_N^+ := \mathbb{R}^{dN}$. An element $Z \in Y_N^+$ is written as $Z = (Z_1, \dots, Z_N)$, where $Z_n \in \mathbb{R}^d$ for $n \in \{1, \dots, N\}$. The restriction operator $R_N^+: Y^+ \rightarrow Y_N^+$ is given by

$$R_N^+ z := (z(t_{N,1}), \dots, z(t_{N,N})).$$

The prolongation operator $P_N^+: Y_N^+ \rightarrow Y^+$ is the discrete Lagrange interpolation operator

$$P_N^+ Z(t) := \sum_{n=1}^N \ell_{N,n}^+(t) Z_n, \quad t \in [0, h],$$

where $\ell_{N,1}^+, \dots, \ell_{N,N}^+$ are the Lagrange coefficients relevant to the nodes of Ω_N^+ . Observe again that

$$R_N^+ P_N^+ = I_{Y_N^+}, \quad P_N^+ R_N^+ = \mathcal{L}_N^+, \quad (4.13)$$

where $\mathcal{L}_N^+: Y^+ \rightarrow Y^+$ is the Lagrange interpolation operator that associates to a function $z \in Y^+$ the $(N-1)$ -degree $\mathbb{R}^d$ -valued polynomial $\mathcal{L}_N^+ w$ such that

$$\mathcal{L}_N^+ z(t_{N,n}) = z(t_{N,n})$$

for $n \in \{1, \dots, N\}$.

Recall from [section 2.1](#) that when not ambiguous (e.g., when applied to an element) the restrictions to subspaces of the above prolongation, restriction and Lagrange interpolation operators are denoted in the same way as the operators themselves.

4.3.3 Discretization of T

The discretization approach we apply here consists in using the prolongation and restriction operators to let T act on the discretized spaces.

Following (4.8) and (4.9), the discretization of indices M and N of the evolution operator T is the finite-dimensional operator $T_{M,N}: Y_M \rightarrow Y_M$ defined as

$$T_{M,N}\Psi := R_M V(P_M \Psi, P_N^+ Z^*)_h,$$

where $Z^* \in Y_N^+$ is a solution of the fixed point equation

$$Z = R_N^+ \mathcal{F}_s V(P_M \Psi, P_N^+ Z) \quad (4.14)$$

for the given $\Psi \in Y_M$. We establish that (4.14) is well posed in [subsection 4.4.2](#).

By virtue of (4.6), the operator $T_{M,N}$ can be rewritten as

$$T_{M,N}\Psi = T_M^{(1)}\Psi + T_{M,N}^{(2)}Z^*,$$

with $T_M^{(1)}: Y_M \rightarrow Y_M$ and $T_{M,N}^{(2)}: Y_N^+ \rightarrow Y_M$ defined as

$$T_M^{(1)}\Psi := R_M(V^- P_M \Psi)_h, \quad T_{M,N}^{(2)}Z := R_M(V^+ P_N^+ Z)_h.$$

Similarly, the fixed point equation (4.14) can be rewritten as

$$(I_{Y_N^+} - U_N^{(2)})Z = U_{M,N}^{(1)}\Psi,$$

with $U_{M,N}^{(1)}: Y_M \rightarrow Y_N^+$ and $U_N^{(2)}: Y_N^+ \rightarrow Y_N^+$ defined as

$$U_{M,N}^{(1)}\Psi := R_N^+ \mathcal{F}_s V^- P_M \Psi, \quad U_N^{(2)}Z := R_N^+ \mathcal{F}_s V^+ P_N^+ Z.$$

Since $I_{Y_N^+} - U_N^{(2)}$ is invertible, the operator $T_{M,N}: Y_M \rightarrow Y_M$ can be eventually reformulated as

$$T_{M,N} = T_M^{(1)} + T_{M,N}^{(2)}(I_{Y_N^+} - U_N^{(2)})^{-1}U_{M,N}^{(1)}.$$

This reformulation simplifies the construction of the matrix representation of $T_{M,N}$; see [appendix A](#) for the sake of the implementation.

4.4 CONVERGENCE ANALYSIS

4.4.1 Additional spaces and assumptions

Consider the subspaces of absolutely continuous (AC) functions $Y_{AC} \subset Y$ and $Y_{AC}^+ \subset Y^+$. Recalling that an AC function has almost everywhere in its domain a derivative which is Lebesgue-integrable, equip these spaces with the norms defined, for $\psi \in Y_{AC}$ and $z \in Y_{AC}^+$, by

$$\begin{aligned} \|\psi\|_{Y_{AC}} &:= \|\psi\|_{L^1([- \tau, 0], \mathbb{R}^d)} + \|\psi'\|_{L^1([- \tau, 0], \mathbb{R}^d)}, \\ \|z\|_{Y_{AC}^+} &:= \|z\|_{L^1([0, h], \mathbb{R}^d)} + \|z'\|_{L^1([0, h], \mathbb{R}^d)}, \end{aligned}$$

with the usual L^1 norms defined as

$$\begin{aligned} \|\psi\|_{L^1([- \tau, 0], \mathbb{R}^d)} &:= \int_{- \tau}^0 |\psi(\sigma)| \, d\sigma, \\ \|z\|_{L^1([0, h], \mathbb{R}^d)} &:= \int_0^h |z(\sigma)| \, d\sigma. \end{aligned}$$

With these choices, all these function spaces are Banach spaces.

The proof of convergence presented in the following sections, requires some hypotheses on the discretization nodes in $[0, h]$ and on the regularity of $\mathcal{F}_s$ and V , in addition to the assumption of [Corollary 4.3](#). They are all referenced individually from the following list where needed:

(H4.1) the meshes $\{\Omega_N^+\}_{N>0}$ are the Chebyshev zeros

$$t_{N,n} := \frac{h}{2} \left(1 - \cos \left(\frac{(2n-1)\pi}{2N} \right) \right), \quad n \in \{1, \dots, N\},$$

(see [Remark 4.14](#));

(H4.2) the hypothesis of [Corollary 4.3](#) holds, namely, $t \mapsto \|L(t)\|_{\mathbb{R}^d \leftarrow Y}$ is bounded;

(H4.3) $\mathcal{F}_s V^+ : Y^+ \rightarrow Y^+$ has range contained in Y_{AC}^+ and $\mathcal{F}_s V^+ : Y^+ \rightarrow Y_{AC}^+$ is bounded;

(H4.4) $\mathcal{F}_s V^- : Y \rightarrow Y^+$ is such that $\mathcal{F}_s V^-(Y_{AC}) \subset Y_{AC}^+$ and $\mathcal{F}_s V^-|_{Y_{AC}} : Y_{AC} \rightarrow Y_{AC}^+$ is bounded.

Choosing the RFDE

$$\begin{aligned} y'(t) = & A(t)y(t) + \sum_{k=1}^p B^{(k)}(t)y(t - \tau_k) \\ & + \sum_{k=1}^p \int_{-\tau_k}^{-\tau_{k-1}} C^{(k)}(t, \theta)y(t + \theta) d\theta \end{aligned} \quad (4.15)$$

as a prototype equation, with $\tau_0 := 0 < \tau_1 < \dots < \tau_p := \tau$, [hypotheses \(H4.3\)](#) and [\(H4.4\)](#) are fulfilled if A and $B^{(k)}$ are absolutely continuous and the assumptions of [Proposition 2.8](#) hold[†] for $C^{(k)}$, with boundedness following from [hypothesis \(H4.2\)](#). The choice of this form for the prototype equation is consistent with the models found in the literature on applications of delay equations (see [section 1.1](#) for some relevant references).

4.4.2 Well-posedness of the collocation equation

The first objective of the convergence proof is to show that the discretized equation [\(4.14\)](#) is well posed.

With reference to [\(4.14\)](#), let $\psi \in Y$ and consider the collocation equation

$$Z = R_N^+ \mathcal{F}_s V(\psi, P_N^+ Z) \quad (4.16)$$

in $Z \in Y_N^+$. The aim of this section is to show that [\(4.16\)](#) has a unique solution and to study its relation to the unique solution $z^* \in Y^+$ of [\(4.9\)](#). Using [\(4.6\)](#), the equations [\(4.9\)](#) and [\(4.16\)](#) can be rewritten, respectively, as

$$(I_{Y^+} - \mathcal{F}_s V^+)z = \mathcal{F}_s V^- \psi$$

and

$$(I_{Y_N^+} - R_N^+ \mathcal{F}_s V^+ P_N^+)Z = R_N^+ \mathcal{F}_s V^- \psi. \quad (4.17)$$

The following preliminary result concerns the operators

$$I_{Y^+} - \mathcal{L}_N^+ \mathcal{F}_s V^+ : Y^+ \rightarrow Y^+, \quad (4.18)$$

and

$$I_{Y_N^+} - R_N^+ \mathcal{F}_s V^+ P_N^+ : Y_N^+ \rightarrow Y_N^+. \quad (4.19)$$

Proposition 4.4. *If the operator [\(4.18\)](#) is invertible, then the operator [\(4.19\)](#) is invertible. Moreover, given $\bar{Z} \in Y_N^+$, the unique solution $\hat{z} \in Y^+$ of*

$$(I_{Y^+} - \mathcal{L}_N^+ \mathcal{F}_s V^+)z = P_N^+ \bar{Z} \quad (4.20)$$

[†] [Proposition 2.8](#) assumes that the integrand function u is integrable: in this case, being u continuous, the assumptions can probably be relaxed.

and the unique solution $\hat{Z} \in Y_N^+$ of

$$(I_{Y_N^+} - R_N^+ \mathcal{F}_s V^+ P_N^+) Z = \bar{Z} \quad (4.21)$$

are related by $\hat{Z} = R_N^+ \hat{z}$ and $\hat{z} = P_N^+ \hat{Z}$.

Proof. Apply [Proposition 2.18](#) with $U := Y^+$, $V := Y_N^+$, $A := \mathcal{F}_s V^+$, $P := P_N^+$, $R := R_N^+$, recalling [\(4.13\)](#). $\square$

As observed above, the equation [\(4.16\)](#) is equivalent to [\(4.17\)](#), hence, by choosing

$$\bar{Z} = R_N^+ \mathcal{F}_s V^- \psi, \quad (4.22)$$

it is equivalent to [\(4.21\)](#). Observe also that thanks to [\(4.13\)](#) the equation

$$z = \mathcal{L}_N^+ \mathcal{F}_s V(\psi, z) \quad (4.23)$$

can be rewritten as

$$(I_{Y^+} - \mathcal{L}_N^+ \mathcal{F}_s V^+) z = \mathcal{L}_N^+ \mathcal{F}_s V^- \psi = P_N^+ R_N^+ \mathcal{F}_s V^- \psi,$$

which is equivalent to [\(4.20\)](#) with the choice [\(4.22\)](#). Thus, by [Proposition 4.4](#), if the operator [\(4.18\)](#) is invertible, then the equation [\(4.16\)](#) has a unique solution $Z^* \in Y_N^+$ such that

$$Z^* = R_N^+ z_N^*, \quad z_N^* = P_N^+ Z^*, \quad (4.24)$$

where $z_N^* \in Y^+$ is the unique solution of [\(4.23\)](#). Note for clarity that [\(4.22\)](#) implies $z_N^* = \hat{z}$ for $\hat{z}$ in [Proposition 4.4](#). So, now we show that under suitable assumptions [\(4.18\)](#) is invertible.

Proposition 4.5. *If hypotheses [\(H4.1\)](#), [\(H4.2\)](#) and [\(H4.3\)](#) hold, then there exists a positive integer N_0 such that, for any $N \geq N_0$, the operator [\(4.18\)](#) is invertible and*

$$\|(I_{Y^+} - \mathcal{L}_N^+ \mathcal{F}_s V^+)^{-1}\|_{Y^+ \leftarrow Y^+} \leq 2 \|(I_{Y^+} - \mathcal{F}_s V^+)^{-1}\|_{Y^+ \leftarrow Y^+}.$$

Moreover, for each $\psi \in Y$, [\(4.23\)](#) has a unique solution $z_N^* \in Y^+$ and

$$\|z_N^* - z^*\|_{Y^+} \leq 2 \|(I_{Y^+} - \mathcal{F}_s V^+)^{-1}\|_{Y^+ \leftarrow Y^+} \|\mathcal{L}_N^+ z^* - z^*\|_{Y^+},$$

where $z^* \in Y^+$ is the unique solution of [\(4.9\)](#).

Proof. In this proof, let $I := I_{Y^+}$. By [Theorem 2.17](#), assuming [hypothesis \[\\(H4.1\\)\]\(#\)](#), if $z \in Y_{AC}^+$, then $\mathcal{L}_N^+ z \rightarrow z$ uniformly as $N \rightarrow +\infty$, i.e.,

$$\|(\mathcal{L}_N^+ - I)z\|_{Y^+} \xrightarrow{N \rightarrow +\infty} 0,$$

and the Banach–Steinhaus theorem ([Theorem 2.20](#)) implies

$$\|(\mathcal{L}_N^+ - I)|_{Y_{AC}^+}\|_{Y^+ \leftarrow Y_{AC}^+} \xrightarrow{N \rightarrow +\infty} 0. \quad (4.25)$$

Assuming [hypothesis \[\\(H4.3\\)\]\(#\)](#), this implies

$$\|(\mathcal{L}_N^+ - I)\mathcal{F}_s V^+\|_{Y^+ \leftarrow Y^+} \leq \|(\mathcal{L}_N^+ - I)|_{Y_{AC}^+}\|_{Y^+ \leftarrow Y_{AC}^+} \|\mathcal{F}_s V^+\|_{Y_{AC}^+ \leftarrow Y^+} \xrightarrow{N \rightarrow +\infty} 0. \quad (4.26)$$

There exists in particular a positive integer N_0 such that, for each integer $N \geq N_0$,

$$\|(\mathcal{L}_N^+ - I)\mathcal{F}_s V^+\|_{Y^+ \leftarrow Y^+} \leq \frac{1}{2\|(I - \mathcal{F}_s V^+)^{-1}\|_{Y^+ \leftarrow Y^+}},$$

i.e.,

$$\|(\mathcal{L}_N^+ - I)\mathcal{F}_s V^+\|_{Y^+ \leftarrow Y^+} \|(I - \mathcal{F}_s V^+)^{-1}\|_{Y^+ \leftarrow Y^+} \leq \frac{1}{2}$$

(recall that $I - \mathcal{F}_s V^+$ is invertible with bounded inverse by virtue of [hypothesis \(H4.2\)](#) and [Corollary 4.3](#)). Considering the operator $I - \mathcal{L}_N^+ \mathcal{F}_s V^+$ as a perturbed version of $I - \mathcal{F}_s V^+$ and writing

$$I - \mathcal{L}_N^+ \mathcal{F}_s V^+ = I - \mathcal{F}_s V^+ - (\mathcal{L}_N^+ - I)\mathcal{F}_s V^+,$$

by [Theorem 2.21](#) for each $N \geq N_0$ the operator $I - \mathcal{L}_N^+ \mathcal{F}_s V^+$ is invertible and

$$\begin{aligned} \|(I - \mathcal{L}_N^+ \mathcal{F}_s V^+)^{-1}\|_{Y^+ \leftarrow Y^+} &\leq \frac{\|(I - \mathcal{F}_s V^+)^{-1}\|_{Y^+ \leftarrow Y^+}}{1 - \|(I - \mathcal{F}_s V^+)^{-1}((\mathcal{L}_N^+ - I)\mathcal{F}_s V^+)\|_{Y^+ \leftarrow Y^+}} \\ &\leq 2\|(I - \mathcal{F}_s V^+)^{-1}\|_{Y^+ \leftarrow Y^+}. \end{aligned}$$

Hence, fixed $\psi \in Y$, [\(4.23\)](#) has a unique solution $z_N^* \in Y^+$. For the same ψ , let $e_N^* \in Y^+$ such that $z_N^* = z^* + e_N^*$, where $z^* \in Y^+$ is the unique solution of [\(4.9\)](#). Then

$$\begin{aligned} z^* + e_N^* &= \mathcal{L}_N^+ \mathcal{F}_s V(\psi, z^* + e_N^*) \\ &= \mathcal{L}_N^+ \mathcal{F}_s V(\psi, z^*) + \mathcal{L}_N^+ \mathcal{F}_s V^+ e_N^* \\ &= \mathcal{L}_N^+ z^* + \mathcal{L}_N^+ \mathcal{F}_s V^+ e_N^* \end{aligned}$$

and

$$(I - \mathcal{L}_N^+ \mathcal{F}_s V^+)e_N^* = (\mathcal{L}_N^+ - I)z^*,$$

completing the proof. $\square$

4.4.3 Convergence of the eigenvalues

In order to prove the convergence of the eigenvalues of $T_{M,N}$ to those of T , we first observe that these operators cannot be compared directly, since they are defined on different spaces, and even more so since the two spaces are of finite and infinite dimension, respectively. In view of this, we first translate the problem of studying the eigenvalues of $T_{M,N}$ on Y_M to that of studying the eigenvalues of finite-rank operators $\hat{T}_{M,N}$ and $\hat{T}_N$ on Y ([Propositions 4.6](#) and [4.7](#)). Then, after restricting $\hat{T}_N$ and T to Y_{AC} while preserving the same spectral properties ([Proposition 4.8](#)), we show in [Proposition 4.10](#) that the restricted $\hat{T}_N$ converges in operator norm to the restricted T and, by applying results from spectral approximation theory [[27](#)] ([Lemma 2.25](#)), we obtain the desired convergence of the eigenvalues of $T_{M,N}$ to the eigenvalues of T ([Proposition 4.11](#) and [Theorem 4.12](#)). Moreover, under some additional hypotheses on the smoothness of the eigenfunctions of T (i.e., on the regularity of the model coefficients), the eigenvalues converge with infinite order.

The first step is thus to introduce the finite-rank operator $\hat{T}_{M,N}$ on Y and show the relation between its spectrum and that of $T_{M,N}$.

Proposition 4.6. *The finite-dimensional operator $T_{M,N}$ has the same nonzero eigenvalues, with the same geometric and partial multiplicities, of the operator*

$$\hat{T}_{M,N} := P_M T_{M,N} R_M: Y \rightarrow Y.$$

Moreover, if $\Psi \in Y_M$ is an eigenvector of $T_{M,N}$ associated with a nonzero eigenvalue μ , then $P_M \Psi \in Y$ is an eigenvector of $\hat{T}_{M,N}$ associated with the same eigenvalue μ .

Proof. Apply [Proposition 2.22](#) with $U := Y$, $V := Y_N$, $A := T_{M,N}$, $P := P_M$, $R := R_M$, recalling [\(4.11\)](#). $\square$

Define the operator $\hat{T}_N: Y \rightarrow Y$ as

$$\hat{T}_N \psi := V(\psi, z_N^*)_h,$$

where $z_N^* \in Y^+$ is the solution of the fixed point equation [\(4.23\)](#), which, under [hypotheses \(H4.1\), \(H4.2\) and \(H4.3\)](#), is unique thanks to [Propositions 4.4 and 4.5](#). Observe that z_N^* is a polynomial. Then, for $\psi \in Y$, by [\(4.24\)](#),

$$\begin{aligned} \hat{T}_{M,N} \psi &= P_M T_{M,N} R_M \psi \\ &= P_M R_M V(P_M R_M \psi, P_N^+ Z^*)_h \\ &= \mathcal{L}_M V(\mathcal{L}_M \psi, z_N^*)_h \\ &= \mathcal{L}_M \hat{T}_N \mathcal{L}_M \psi, \end{aligned}$$

where $Z^* \in Y_N^+$ and $z_N^* \in Y^+$ are the solutions, respectively, of [\(4.14\)](#) applied to $\Psi = R_M \psi$ and of [\(4.23\)](#) with $\mathcal{L}_M \psi$ replacing ψ . These solutions are unique under [hypotheses \(H4.1\), \(H4.2\) and \(H4.3\)](#), thanks again to [Propositions 4.4 and 4.5](#).

Now we show the relation between the spectra of $\hat{T}_{M,N}$ and $\hat{T}_N$.

Proposition 4.7. *Assume that [hypotheses \(H4.1\), \(H4.2\) and \(H4.3\)](#) hold and let $M \geq N \geq N_0$, with N_0 given by [Proposition 4.5](#). Then the operator $\hat{T}_{M,N}$ has the same nonzero eigenvalues, with the same geometric and partial multiplicities and associated eigenvectors, of the operator $\hat{T}_N$.*

Proof. Denote by Π_r and Π_r^+ the subspaces of polynomials of degree r of Y and Y^+ , respectively. Note that $z_N^* \in \Pi_{N-1}^+$.

If $h \geq \tau$, for all $\psi \in Y$, $\hat{T}_N \psi = V(\psi, z_N^*)_h \in \Pi_N$. Thus both $\hat{T}_N$ and $\hat{T}_{M,N} = \mathcal{L}_M \hat{T}_N \mathcal{L}_M$ have range contained in Π_M , being $M \geq N$. By [Proposition 2.23](#) and [Remark 2.24](#), $\hat{T}_N$ and $\hat{T}_{M,N}$ have the same nonzero eigenvalues, with the same geometric and partial multiplicities and associated eigenvectors, as their restrictions to Π_M . Observing that

$$\hat{T}_{M,N}|_{\Pi_M} = \mathcal{L}_M \hat{T}_N \mathcal{L}_M|_{\Pi_M} = \hat{T}_N|_{\Pi_M},$$

the thesis follows.

Consider now the case $h < \tau$. Denote by Π_r^{pw} the subspace of piecewise polynomials of degree r of Y on the intervals $[\theta^{(q+1)}, \theta^{(q)}]$, for $q = 0, \dots, Q-1$. For all $\psi \in \Pi_M^{\text{pw}}$, $\hat{T}_N \psi = V(\psi, z_N^*)_h \in \Pi_M^{\text{pw}}$. Let $\mu \neq 0$, $\psi \in Y$ and $\bar{\psi} \in \Pi_M^{\text{pw}}$ such that

$$(\mu I_Y - \hat{T}_N) \psi = \mu \psi - V(\psi, z_N^*)_h = \bar{\psi}.$$

This equation can be rewritten as

$$\begin{cases} \mu\psi(\theta) = \psi(0) + \int_0^{h+\theta} z_N^*(\sigma) d\sigma + \bar{\psi}(\theta), & \text{if } \theta \in [-h, 0], \\ \mu\psi(\theta) = \psi(h+\theta) + \bar{\psi}(\theta), & \text{if } \theta \in [-\tau, -h]. \end{cases}$$

From the first equation, ψ restricted to $[-h, 0]$ is a polynomial of degree M . From the second equation it is easy to show that $\psi \in \Pi_M^{\text{pw}}$ by induction on the intervals $[\theta^{(q+1)}, \theta^{(q)}]$, for $q = 1, \dots, Q-1$. Hence, by [Proposition 2.23](#), $\hat{T}_N$ has the same nonzero eigenvalues, with the same geometric and partial multiplicities and associated eigenvectors, as its restriction to Π_M^{pw} . The same holds for $\hat{T}_{M,N} = \mathcal{L}_M \hat{T}_N \mathcal{L}_M$ by [Proposition 2.23](#) and [Remark 2.24](#) since its range is contained in Π_M^{pw} . The thesis follows by observing that

$$\hat{T}_{M,N} \upharpoonright_{\Pi_M^{\text{pw}}} = \mathcal{L}_M \hat{T}_N \mathcal{L}_M \upharpoonright_{\Pi_M^{\text{pw}}} = \hat{T}_N \upharpoonright_{\Pi_M^{\text{pw}}}. \quad \square$$

Thanks to [Proposition 2.23](#), the spectral properties of T and $\hat{T}_N$ are preserved when they are restricted to the subspace of Y of absolutely continuous functions, a step required in order to achieve the desired convergence properties of Lagrange interpolation.

Proposition 4.8. *The operators T and $\hat{T}_N$ have the same nonzero eigenvalues, with the same geometric and partial multiplicities and associated eigenvectors, as their restrictions to Y_{AC} .*

Proof. First observe that if $\psi \in Y_{\text{AC}}$ and $z \in Y^+$ then $V(\psi, z)_h \in Y_{\text{AC}}$, hence $T(Y_{\text{AC}}) \subset Y_{\text{AC}}$ and $\hat{T}_N(Y_{\text{AC}}) \subset Y_{\text{AC}}$. Moreover, if $\psi \in Y$, $z \in Y^+$ and $h \geq \tau$, then $V(\psi, z)_h \in Y_{\text{AC}}$. Let $\mu \neq 0$, $\psi \in Y$ and $\bar{\psi} \in Y_{\text{AC}}$ such that

$$(\mu I_Y - T)\psi = \mu\psi - V(\psi, z^*)_h = \bar{\psi}, \quad (4.27)$$

where z^* is the solution of (4.9). If $h \geq \tau$ then (4.27) implies that $\psi \in Y_{\text{AC}}$. If $h < \tau$ then (4.27) can be rewritten as

$$\begin{cases} \mu\psi(\theta) = \psi(0) + \int_0^{h+\theta} z^*(\sigma) d\sigma + \bar{\psi}(\theta), & \text{if } \theta \in (-h, 0], \\ \mu\psi(\theta) = \psi(h+\theta) + \bar{\psi}(\theta), & \text{if } \theta \in [-\tau, -h]. \end{cases}$$

From the first equation, ψ restricted to $[-h, 0]$ is AC. From the second equation it is easy to show that $\psi \in Y_{\text{AC}}$ by induction on the intervals $[\theta^{(q+1)}, \theta^{(q)}]$, for $q = 1, \dots, Q-1$. The same argument holds for $\hat{T}_N$. The thesis follows by [Proposition 2.23](#). $\square$

Below we prove the norm convergence of $\hat{T}_N$ to T when both are restricted to Y_{AC} , which is the key step to obtain the main result of this chapter. First we need to extend the results of [Corollary 4.3](#) to $(I_{Y^+} - \mathcal{F}_s V^+) \upharpoonright_{Y_{\text{AC}}^+}$ in the following lemma.

Lemma 4.9. *If hypotheses (H4.2) and (H4.3) hold, then $(I_{Y^+} - \mathcal{F}_s V^+) \upharpoonright_{Y_{\text{AC}}^+}$ is invertible with bounded inverse.*

Proof. Since $I_{Y^+} - \mathcal{F}_s V^+$ is invertible with bounded inverse by virtue of [hypothesis \(H4.2\)](#) and [Corollary 4.3](#), given $f \in Y_{AC}^+$ the equation $(I_{Y^+} - \mathcal{F}_s V^+)z = f$ has a unique solution $z \in Y^+$, which by [hypothesis \(H4.3\)](#) is in Y_{AC}^+ . Hence, the operator $(I_{Y^+} - \mathcal{F}_s V^+) \upharpoonright_{Y_{AC}^+}$ is invertible. It is also bounded, since by [Theorem 2.6](#) there exists $C > 0$ such that $\|\cdot\|_{Y^+} \leq C\|\cdot\|_{Y_{AC}^+}$ in Y_{AC}^+ , which implies

$$\|\mathcal{F}_s V^+ \upharpoonright_{Y_{AC}^+}\|_{Y_{AC}^+ \leftarrow Y_{AC}^+} \leq C \|\mathcal{F}_s V^+\|_{Y_{AC}^+ \leftarrow Y^+}.$$

The bounded inverse theorem ([Theorem 2.19](#)) completes the proof. $\square$

Proposition 4.10. *If [hypotheses \(H4.1\)](#), [\(H4.2\)](#), [\(H4.3\)](#) and [\(H4.4\)](#) hold, then*

$$\left\| \hat{T}_N \upharpoonright_{Y_{AC}} - T \upharpoonright_{Y_{AC}} \right\|_{Y_{AC} \leftarrow Y_{AC}} \xrightarrow{N \rightarrow +\infty} 0.$$

Proof. Let $\psi \in Y_{AC}$ and let z^* and z_N^* be the solutions of the fixed point equations [\(4.9\)](#) and [\(4.23\)](#), respectively. Then

$$(\hat{T}_N - T)\psi = V(\psi, z_N^*)_h - V(\psi, z^*)_h = V^+(z_N^* - z^*)_h.$$

Assuming [hypotheses \(H4.3\)](#) and [\(H4.4\)](#) and recalling that

$$z^* = \mathcal{F}_s V^+ z^* + \mathcal{F}_s V^- \psi,$$

it is clear that $z^* \in Y_{AC}^+$. Assuming also [hypotheses \(H4.1\)](#) and [\(H4.2\)](#), by [Proposition 4.5](#) there exists a positive integer N_0 such that, for any $N \geq N_0$,

$$\begin{aligned} \|(\hat{T}_N - T)\psi\|_{Y_{AC}} &= \|V^+(z_N^* - z^*)_h\|_{Y_{AC}} \\ &\leq \left\| \int_0^\cdot (z_N^* - z^*)(\sigma) d\sigma \right\|_{Y_{AC}^+} \\ &\leq h \left(1 + \frac{h}{2}\right) \|z_N^* - z^*\|_{Y^+} \\ &\leq 2h \left(1 + \frac{h}{2}\right) \|(I_{Y^+} - \mathcal{F}_s V^+)^{-1}\|_{Y^+ \leftarrow Y^+} \|\mathcal{L}_N^+ z^* - z^*\|_{Y^+} \\ &\leq 2h \left(1 + \frac{h}{2}\right) \|(I_{Y^+} - \mathcal{F}_s V^+)^{-1}\|_{Y^+ \leftarrow Y^+} \\ &\quad \|(\mathcal{L}_N^+ - I_{Y^+}) \upharpoonright_{Y_{AC}^+}\|_{Y^+ \leftarrow Y_{AC}^+} \|z^*\|_{Y_{AC}^+}. \end{aligned} \tag{4.28}$$

Moreover, thanks to [Lemma 4.9](#) and [hypothesis \(H4.4\)](#),

$$\|z^*\|_{Y_{AC}^+} \leq \|((I_{Y^+} - \mathcal{F}_s V^+) \upharpoonright_{Y_{AC}^+})^{-1}\|_{Y_{AC}^+ \leftarrow Y_{AC}^+} \|\mathcal{F}_s V^- \upharpoonright_{Y_{AC}}\|_{Y_{AC}^+ \leftarrow Y_{AC}} \|\psi\|_{Y_{AC}}. \tag{4.29}$$

The thesis follows by [\(4.25\)](#). $\square$

The final convergence result is obtained thanks to results from spectral approximation theory, namely the ones summarized in [Lemma 2.25](#), and classic results in interpolation theory.

Proposition 4.11. *Assume that [hypotheses \(H4.1\)](#), [\(H4.2\)](#), [\(H4.3\)](#) and [\(H4.4\)](#) hold. If $\mu \in \mathbb{C} \setminus \{0\}$ is an eigenvalue of $T \upharpoonright_{Y_{AC}}$ with finite algebraic multiplicity ν and ascent l , and Δ is a neighborhood of μ such that μ is the only eigenvalue of $T \upharpoonright_{Y_{AC}}$ in*

Δ , then there exists a positive integer $N_1 \geq N_0$, with N_0 given by [Proposition 4.5](#), such that, for any $N \geq N_1$, $\hat{T}_N|_{Y_{AC}}$ has in Δ exactly ν eigenvalues $\mu_{N,j}$, $j \in \{1, \dots, \nu\}$, counting their multiplicities. Moreover, if for each $\psi \in \mathcal{E}_\mu$, where $\mathcal{E}_\mu$ is the generalized eigenspace of $T|_{Y_{AC}}$ associated with μ , the function z^* that solves (4.9) is of class C^p , with $p \geq 1$, then

$$\max_{j \in \{1, \dots, \nu\}} |\mu_{N,j} - \mu| = o(N^{\frac{1-p}{1}}).$$

Proof. By [Proposition 4.10](#),

$$\|(\hat{T}_N - T)|_{Y_{AC}}\|_{Y_{AC} \leftarrow Y_{AC}} \xrightarrow{N \rightarrow +\infty} 0.$$

The first part of the thesis is obtained by applying [Lemma 2.25](#). From the same [Lemma 2.25](#), (2.16) follows with

$$\epsilon_N := \|(\hat{T}_N - T)|_{\mathcal{E}_\mu}\|_{Y_{AC} \leftarrow \mathcal{E}_\mu}$$

and $\mathcal{E}_\mu$ the generalized eigenspace of μ equipped with the norm of Y_{AC} restricted to $\mathcal{E}_\mu$.

Let $\psi_1, \dots, \psi_\nu$ be a basis of $\mathcal{E}_\mu$. An element ψ of $\mathcal{E}_\mu$ can be written as

$$\psi = \sum_{j=1}^{\nu} \alpha_j(\psi) \psi_j,$$

with $\alpha_j(\psi) \in \mathbb{C}$, for $j \in \{1, \dots, \nu\}$, hence

$$\|(\hat{T}_N - T)\psi\|_{Y_{AC}} \leq \max_{j \in \{1, \dots, \nu\}} |\alpha_j(\psi)| \sum_{j=1}^{\nu} \|(\hat{T}_N - T)\psi_j\|_{Y_{AC}}.$$

The function

$$\psi \mapsto \max_{j \in \{1, \dots, \nu\}} |\alpha_j(\psi)|$$

is a norm on $\mathcal{E}_\mu$, so it is equivalent to the norm of Y_{AC} restricted to $\mathcal{E}_\mu$. Thus, there exists a positive constant c independent of ψ such that

$$\max_{j \in \{1, \dots, \nu\}} |\alpha_j(\psi)| \leq c \|\psi\|_{Y_{AC}}$$

and

$$\epsilon_N = \|(\hat{T}_N - T)|_{\mathcal{E}_\mu}\|_{Y_{AC} \leftarrow \mathcal{E}_\mu} \leq c \sum_{j=1}^{\nu} \|(\hat{T}_N - T)\psi_j\|_{Y_{AC}}.$$

Let $j \in \{1, \dots, \nu\}$. As seen in [Proposition 4.10](#),

$$\|(\hat{T}_N - T)\psi_j\|_{Y_{AC}} \leq 2h \left(1 + \frac{h}{2}\right) \|(I_{Y^+} - \mathcal{F}_s V^+)^{-1}\|_{Y^+ \leftarrow Y^+} \|(\mathcal{L}_N^+ - I)z_j^*\|_{Y^+},$$

where z_j^* is the solution of (4.9) associated with ψ_j . Now, by [Theorem 2.14](#) and [Corollary 2.12](#), since z_j^* is of class C^p , the bound

$$\begin{aligned} \|(\mathcal{L}_N^+ - I)z_j^*\|_{Y^+} &\leq (1 + \Lambda_N) E_{N-1}(z_j^*) \\ &\leq (1 + \Lambda_N) \frac{6^{p+1} e^p}{1+p} \left(\frac{h}{2}\right)^p \frac{1}{(N-1)^p} \omega\left(\frac{h}{2(N-1-p)}\right) \end{aligned}$$

holds, where Λ_N is the Lebesgue constant for Ω_N^+ , $E_{N-1}(\cdot)$ is the best uniform approximation error and $\omega(\cdot)$ is the modulus of continuity of $(z_j^*)^{(p)}$ on $[0, h]$. Since [hypothesis \(H4.1\)](#) is assumed, by [Theorem 2.15](#), $\Lambda_N = o(N)$. Hence, $\epsilon_N = o(N^{1-p})$ and the thesis follows immediately. $\square$

Theorem 4.12. Assume that *hypotheses* (H4.1), (H4.2), (H4.3) and (H4.4) hold. If $\mu \in \mathbb{C} \setminus \{0\}$ is an eigenvalue of T with finite algebraic multiplicity v and ascent l , and Δ is a neighborhood of μ such that μ is the only eigenvalue of T in Δ , then there exists a positive integer $N_1 \geq N_0$, with N_0 given by Proposition 4.5, such that, for any $N \geq N_1$ and any $M \geq N$, $T_{M,N}$ has in Δ exactly v eigenvalues $\mu_{M,N,j}$, $j \in \{1, \dots, v\}$, counting their multiplicities. Moreover, if for each $\psi \in \mathcal{E}_\mu$, where $\mathcal{E}_\mu$ is the generalized eigenspace of T associated with μ , the function z^* that solves (4.9) is of class C^p , with $p \geq 1$, then

$$\max_{j \in \{1, \dots, v\}} |\mu_{M,N,j} - \mu| = o(N^{\frac{1-p}{T}}). \quad (4.30)$$

Proof. If $M \geq N \geq N_0$, by Propositions 4.6 and 4.7 the operators $T_{M,N}$, $\hat{T}_{M,N}$ and $\hat{T}_N$ have the same nonzero eigenvalues, with the same geometric and partial multiplicities and associated eigenvectors. By Proposition 4.8, the operators T and $\hat{T}_N$ have the same nonzero eigenvalues, with the same geometric and partial multiplicities and associated eigenvectors, as their restrictions to Y_{AC} . The thesis follows by Proposition 4.11. $\square$

Remark 4.13. Theorem 4.12 ensures that if the solutions of the IVP corresponding to an initial state in the generalized eigenspace are smooth, then an infinite order of convergence can be achieved. It is easy to show that this is the case for RFDEs. Indeed if the initial value ψ is an eigenfunction of T , then the solution must be smooth, since the solution is of class C^k on $[(k-1)\tau, k\tau]$, but states at the times jh are multiples of ψ . By induction on the rank, solutions are smooth also for (linear combinations of) generalized eigenfunctions as initial value.

The infinite convergence order is a key computational feature. It allows to obtain accurate results with low dimensions of the discretized problem and therefore low computational costs, and this renders performing robust analyses of stability and bifurcations attainable. $\triangleleft$

Remark 4.14. Nodes other than those required by hypothesis (H4.1) may be used. Indeed, they are only asked to satisfy $\Lambda_N = o(N)$ and the thesis of Theorem 2.17. Anyway, here we assume hypothesis (H4.1) since these are the nodes we actually use in implementing the method (see section A.2). $\triangleleft$

Remark 4.15. In general, it may not be possible to compute exactly the integrals in (4.15). If this is the case, an approximation $\tilde{\mathcal{F}}_s$ of $\mathcal{F}_s$ must be used, leading to a further contribution in the final error, so that, for instance, (4.30) becomes

$$\max_{j \in \{1, \dots, v\}} |\mu_{M,N,j} - \mu| = o(N^{\frac{1-p}{T}} + \text{TOL}).$$

If the quadrature formula is chosen in such a way that TOL decreases as M and N increase, then the eigenvalues of $T_{M,N}$ still converge to those of T , otherwise TOL acts as a “barrier” on the accuracy that can be achieved.

The quadrature formula can be chosen in such a way that also TOL decreases with infinite order, e.g., if the Clenshaw–Curtis quadrature formula is used with a number of nodes of the order of M or N . See section A.2 for details on this choice for the current MATLAB/Octave implementation. $\triangleleft$

5

RENEWAL EQUATIONS

In this chapter we extend the method of [15, 16] presented in [chapter 4](#) to renewal equations (REs). This is the main novel contribution of this thesis, along with the content of [chapter 6](#).

Most of the content of this chapter is part of a paper by the author and D. Breda which is currently under revision.

The different type of equations has several consequences, reflecting the fact that RFDEs specify the derivative of the solution at a given time, while REs specify the value of the function itself.

First of all, the state space is no longer a space of continuous functions, but instead it is a space of L^1 functions. This is a natural choice when working with REs, since in general there might be a discontinuity in the solution at the initial time even if the initial state is continuous. Moreover, it is consistent with the usual assumptions in the literature on applications of delay equations. Consider, e.g., the *Daphnia* model described in [section 1.1](#): there, the quantity b described by the RE is the birth rate: as observed in [34], although the rate itself may be unbounded, it must have a finite integral (i.e., the number of individuals). See [section 1.1](#) for more references to applications.

Another important difference between the two versions of the method lies in the definition of the operator V in (4.5), which indeed is the operator that captures the rule to construct the solution from the data obtained from the equation.

Finally, the right-hand side of REs, thanks to the integral term, under suitable hypotheses on the integration kernel exhibits a regularization effect, meaning that applying $\mathcal{F}_s$ to an L^1 function produces a continuous function. A similar regularization effect is absent in general in RFDEs. Indeed, we show that the restriction of the state space, which was necessary for RFDEs in order to prove the convergence, is not required in the case of REs.

The above and other differences will be highlighted in the exposition that follows and also in [chapters 6](#) and [7](#).

5.1 EVOLUTION OPERATORS FOR LINEAR REs

Let $d \in \mathbb{N}$ and $\tau \in \mathbb{R}$, both positive, and consider the function space

$$X := L^1([-\tau, 0], \mathbb{R}^d)$$

equipped with the usual L^1 norm

$$\|\psi\|_X := \int_{-\tau}^0 |\psi(\theta)| \, d\theta. \quad (5.1)$$

A linear RE with finite delay is a relation of the form

$$x(t) = \int_{-\tau}^0 C(t, \theta) x_t(\theta) d\theta, \quad t \in \mathbb{R}, \quad (5.2)$$

where x_t is defined as in (2.1) and $C: \mathbb{R} \times [-\tau, 0] \rightarrow \mathbb{R}^{d \times d}$ is a measurable function. The numbers d and τ are, respectively, the dimension of the equation and the maximum delay, while X is the state space and $x_t \in X$ is the state at time t .

For $s \in \mathbb{R}$ and $\varphi \in X$, the Cauchy problem for (5.2) is defined as

$$\begin{cases} x(t) = \int_{-\tau}^0 C(t, \theta) x_t(\theta) d\theta, & t > s, \\ x_s = \varphi. \end{cases} \quad (5.3)$$

A function x is a solution of (5.3) on $[s - \tau, s + t_f)$ if there exists $t_f > 0$ such that $x \in L^1_{\text{loc}}([s - \tau, s + t_f], \mathbb{R}^d)$, $x_s = \varphi$ and for each $t \in [s, s + t_f)$ $x(t)$ satisfies (5.2). The final time t_f may be $+\infty$. To emphasize the dependence of solutions on both the initial time s and the initial function φ , a solution $x(\cdot)$ of (5.3) is sometimes denoted as $x(\cdot; s, \varphi)$.

For $t \in [s, s + \tau]$, by setting $\tilde{x}(t - s) := x(t)$, the Cauchy problem (5.3) corresponds to the Volterra integral equation (VIE) of the second kind

$$\tilde{x}(r) = \int_0^r K(r, \sigma) \tilde{x}(\sigma) d\sigma + f(r), \quad r \in [0, \tau], \quad (5.4)$$

for

$$K(r, \sigma) := C(s + r, \sigma - r) \quad (5.5)$$

and

$$f(r) := \int_{r-\tau}^0 K(r, \sigma) \varphi(\sigma) d\sigma.$$

Theorem 5.1. *If the interval $[0, \tau]$ can be partitioned into finitely many subintervals $J_1, \dots, J_n$ such that*

$$\text{ess sup}_{\sigma \in J_i} \int_{J_i} |K(r, \sigma)| dr < 1, \quad i \in \{1, \dots, n\},$$

then, given $f \in L^1([0, \tau], \mathbb{R}^d)$, the VIE (5.4) has a unique solution $\tilde{x}$ in the space $L^1([0, \tau], \mathbb{R}^d)$.

Proof. It follows directly from Theorems 2.26 and 2.27. $\square$

Corollary 5.2. *If the interval $[0, \tau]$ can be partitioned into finitely many subintervals $J_1, \dots, J_n$ such that, for any $s \in \mathbb{R}$,*

$$\text{ess sup}_{\sigma \in J_i} \int_{J_i} |C(s + r, \sigma - r)| dr < 1, \quad i \in \{1, \dots, n\}, \quad (5.6)$$

then the Cauchy problem (5.3) admits a unique solution on $[s - \tau, s + t_f)$.

Proof. If $t_f \leq \tau$, this follows from Theorem 5.1 via (5.5). Otherwise, the same argument can be repeated on $[\tau, 2\tau]$, $[2\tau, 3\tau]$ and so on. $\square$

By [Corollary 5.2](#), assuming the required regularity of the kernel C , the solution of the Cauchy problem [\(5.3\)](#) exists uniquely and bounded in $L^1([s - \tau, s + t_f], \mathbb{R}^d)$. Moreover, a reasoning on the lines of Bellman's method of steps [\[5, 7\]](#) (see also [\[6\]](#) and [\[4, section 3.4\]](#) for similar arguments, and [\[20, section 4.1.2\]](#) for VIEs) allows to extend the solution to any $t > s$, by working successively on $[s + \tau, s + 2\tau]$, $[s + 2\tau, s + 3\tau]$ and so on, yielding a unique solution x on $[s - \tau, +\infty)$. This allows us to define the family $\{T(t, s)\}_{(t, s) \in \Delta}$ of evolution operators

$$T(t, s): X \rightarrow X, \quad T(t, s)\varphi := x_t(\cdot; s, \psi), \quad (5.7)$$

where

$$\Delta := \{(t, s) \in \mathbb{R}^2 \mid -\infty \leq s \leq t \leq +\infty\}.$$

The following proposition is a consequence of [Theorem 3.1](#) and of the correspondence between solutions of the linear initial value problem and the relevant abstract equation (see [section 3.3](#)). It is also possible to obtain a direct proof as for the analogous [Proposition 4.2](#).

Proposition 5.3. *The family of evolution operators $\{T(t, s)\}_{(t, s) \in \Delta}$ defined in [\(5.7\)](#) is a strongly continuous evolutionary system.*

Let $s \in \mathbb{R}$ and $h \geq 0$ and consider the evolution operator

$$T := T(s + h, s).$$

The aim of this chapter is to approximate the spectrum of T by computing with standard techniques the eigenvalues of a finite-dimensional approximation of T obtained via pseudospectral collocation, as described in [section 5.3](#).

Recall again from [sections 1.2](#) and [4.1](#) that this allows to study the stability of equilibria and periodic solutions, and can lead to the application to Lyapunov exponents.

Remark 5.4. Compared to the general linear RFDE [\(4.2\)](#), the chosen prototype linear RE [\(5.2\)](#) has an apparently special form, but this does not cause a loss of generality. Indeed, thanks to the Riesz representation theorem for L^1 ([Theorem 2.30](#)), every linear retarded functional equation of the type $x(t) = L(t)x_t$ can be written in the form [\(5.2\)](#). However, not all of them satisfy the assumptions of [Corollary 5.2](#). Being the uniqueness of solutions essential to our problem, we exclude these equations from this exposition.

One might be tempted to consider a prototype linear RE similar to [\(4.15\)](#), but terms involving the value of the solution at given points are not well defined in terms of L^1 functions. Even in the settings where this kind of terms can be defined, such equations do not ensure the regularization of solutions as it happens for the analogous RFDEs, and this is fundamental for the convergence of the numerical method. Moreover, the equations might be of neutral type, a case out of the scope of this work because of the additional theoretical challenges that it poses. $\triangleleft$

Remark 5.5. Comparing again [\(5.2\)](#) with [\(4.15\)](#), in many applications the function $C(t, \theta)$ (is continuous in t and) has a finite number of discontinuities in θ . Hence [\(5.2\)](#) may often be written in the form

$$x(t) = \sum_{k=1}^p \int_{-\tau_k}^{-\tau_{k-1}} C^{(k)}(t, \theta) x(t + \theta) d\theta \quad (5.8)$$

with $\tau_0 := 0 < \tau_1 < \dots < \tau_p := \tau$ and $C^{(k)}(t, \theta)$ continuous in θ . This choice agrees, for instance, with the literature on physiologically- and age-structured populations (where discontinuities are due, e.g., to different behavior of juveniles and adults); see [section 1.1](#) for some relevant references. $\triangleleft$

5.2 REFORMULATION OF T

As in [section 4.2](#), we base the discretization of T on a suitable reformulation, which follows the same philosophy as for RFDEs.

Define the function spaces

$$X^+ := L^1([0, h], \mathbb{R}^d), \quad X^\pm := L^1([-\tau, h], \mathbb{R}^d),$$

equipped with the corresponding L^1 norms denoted, respectively, by $\|\cdot\|_{X^+}$ and $\|\cdot\|_{X^\pm}$.

As in the RFDE case, the operator V captures the rule to construct a solution of (5.3), while the operator $\mathcal{F}_s$ applies the right-hand side functional to its argument after a time shift. Although for both operators the exact definition differs between the RFDE and the RE case, the change in V is more substantial, since it reflects the different kind of equation. Indeed for RE the data obtained from the equation is the solution itself.

Define the operator $V: X \times X^+ \rightarrow X^\pm$ as

$$V(\varphi, w)(t) := \begin{cases} w(t), & t \in (0, h], \\ \varphi(t), & t \in [-\tau, 0]. \end{cases} \quad (5.9)$$

Let $V^-: X \rightarrow X^\pm$ and $V^+: X^+ \rightarrow X^\pm$ be given, respectively, by $V^- \varphi := V(\varphi, 0_{X^+})$ and $V^+ w := V(0_X, w)$. Observe that

$$V(\varphi, w) = V^- \varphi + V^+ w. \quad (5.10)$$

Note also that $V(\varphi, w)(t)$ can have a discontinuity in 0 even when φ and w are continuous but $\varphi(0) \neq w(0)$. This is another important difference with respect to the RFDE case of [chapter 4](#), which calls later on for special attention to discontinuities and to the role of 0, both in the theoretical treatment of the numerical method and in its implementation.

Remark 5.6. The choice of including $t = 0$ in the past in (5.9), as well as in (5.3), is common for REs modeling, e.g., structured populations [34, 36]. From the theoretical point of view, it does not make any difference, since X consists of equivalence classes of functions coinciding almost everywhere. From the interpretative point of view, it can be motivated by the consideration that although the actual value $\varphi(0)$ is not well defined, being φ in L^1 , it is reasonable to define the solution as coinciding with the initial function φ of the problem on the whole domain of φ . Moreover, from the implementation point of view, numerical tests performed including $t = 0$ in the past or in the future show that either choice gives the same results, with the only (obvious) requirement to be consistent throughout the code. $\triangleleft$

Define the operator $\mathcal{F}_s: X^\pm \rightarrow X^+$ as

$$\mathcal{F}_s u(t) := \int_{-\tau}^0 C(s+t, \theta) u(t+\theta) d\theta, \quad t \in [0, h]. \quad (5.11)$$

As noted above, observe that, although different in the exact formulation, there is no conceptual difference between (4.7) and (5.11).

The evolution operator T can be reformulated as

$$T\varphi = V(\varphi, w^*)_h, \quad (5.12)$$

where $w^* \in X^+$ is the solution of the fixed point equation

$$w = \mathcal{F}_s V(\varphi, w), \quad (5.13)$$

which exists uniquely and bounded thanks to [Corollary 5.7](#) below. Recall that in (5.12) the subscript h is used according to [Definition 2.1](#), hence

$$V(\varphi, w^*)_h(\theta) = V(\varphi, w^*)(h + \theta)$$

for $\theta \in [-\tau, 0]$.

Similarly to [Corollary 5.7](#), the next result follows from [Corollary 5.2](#), stating the existence and uniqueness of solutions in a useful form for the convergence proof.

Corollary 5.7. *If the interval $[0, \tau]$ can be partitioned into finitely many subintervals $J_1, \dots, J_n$ such that, for any $s \in \mathbb{R}$, (5.6) holds, then the operator $I_{X^+} - \mathcal{F}_s V^+$ is invertible with bounded inverse and (5.13) admits a unique solution in X^+ .*

Proof. Given $f \in X^+$, the equation $(I_{X^+} - \mathcal{F}_s V^+)w = f$ has a unique solution $w \in X^+$ if and only if the initial value problem

$$\begin{cases} w(t) = \int_{-\tau}^0 C(s+t, \theta) w(t+\theta) d\theta + f(t), & t \in (0, h], \\ w_0 = 0 \in X, \end{cases}$$

has a unique solution in $X^\pm$, with the two solutions coinciding on $[0, h]$. This follows from [Corollary 5.2](#). So $I_{X^+} - \mathcal{F}_s V^+$ is invertible and bounded and the bounded inverse theorem ([Theorem 2.19](#)) completes the proof. $\square$

5.3 DISCRETIZATION

Let M and N be positive integers, referred to as discretization indices. Define the partitions of the time intervals $[-\tau, 0]$ and $[0, h]$ as in [subsection 4.3.1](#).

As for the function spaces, we proceed as in [subsection 4.3.2](#) to define the discretized spaces X_M and X_N^+ and the operators $P_M, R_M, \mathcal{L}_M, P_N^+, R_N^+$ and $\mathcal{L}_N^+$ with one important difference. Since an L^1 function is an equivalence class of functions equal almost everywhere, values in specific points are not well defined, so it does not seem reasonable to define the restriction operators on the whole spaces X and X^+ . Hence, we define them, and thus also the Lagrange interpolation operators, on subspaces $\tilde{X} \subset X$ and $\tilde{X}^+ \subset X^+$ regular enough to make point-wise evaluation meaningful.

Indeed, this is amply justified. First of all, it is clear from the following sections that the restriction and interpolation operators are actually applied only to continuous functions or polynomials (or their piecewise counterparts if $h < \tau$). Moreover, our interest is in the eigenfunctions of the evolution operator (see [Theorem 5.16](#) below), which are expected to be sufficiently regular (see relevant comments in [chapter 9](#)). As a last argument, ultimately, the numerical method is applied to finite-dimensional vectors, which bear no notion of the function from which they are derived.

The equalities (4.11) and (4.13) hold also in this setting, observing that it is surely reasonable to assume the images of the prolongation operators, and hence of the Lagrange interpolation operators, to be contained in $\tilde{X}$ and $\tilde{X}^+$. As in [subsection 4.3.2](#) when not ambiguous the restrictions to subspaces of the prolongation, restriction and Lagrange interpolation operators are denoted in the same way as the operators themselves.

Finally, discretize the operator T as $T_{M,N}: X_M \rightarrow X_M$ according to [subsection 4.3.3](#), obtaining

$$T_{M,N}\Phi := R_M V(P_M \Phi, P_N^+ W^*)_h,$$

where $W^* \in X_N^+$ is a solution of the fixed point equation

$$W = R_N^+ \mathcal{F}_s V(P_M \Phi, P_N^+ W) \quad (5.14)$$

for the given $\Phi \in X_M$, along with the further reformulation

$$T_{M,N} = T_M^{(1)} + T_{M,N}^{(2)} (I_{X_N^+} - U_N^{(2)})^{-1} U_{M,N}^{(1)}, \quad (5.15)$$

with definitions for the various finite-dimensional operators analogous to the ones in [subsection 4.3.3](#). We establish that (5.14) is well posed in [subsection 5.4.2](#). In [appendix A](#) the reformulation (5.15) is exploited to construct the matrix representation of $T_{M,N}$.

5.4 CONVERGENCE ANALYSIS

The convergence analysis follows the lines of [section 4.4](#), with the unavoidable differences that will be highlighted in the exposition.

5.4.1 Additional spaces and assumptions

Consider the space of continuous functions

$$X_C^+ := C([0, h], \mathbb{R}^d) \subset X^+$$

equipped with the uniform norm, denoted by $\|\cdot\|_{X_C^+}$. If $h \geq \tau$ consider also

$$X_C := C([- \tau, 0], \mathbb{R}^d) \subset X$$

equipped with the uniform norm, denoted by $\|\cdot\|_{X_C}$. If $h < \tau$, instead, define the space $X_C \subset X$ as

$$X_C := \{\varphi \in X \mid \varphi|_{(\theta^{(q+1)}, \theta^{(q)})} \in C((\theta^{(q+1)}, \theta^{(q)}), \mathbb{R}^d), q \in \{0, \dots, Q-1\}\}$$

and the one-sided limits at $\theta^{(q)}$ exist finite, $q \in \{0, \dots, Q\}$,

equipped with the same norm $\|\cdot\|_{X_C}$. With these choices, all these function spaces are Banach spaces.

Remark 5.8. Observe that X_C and X_C^+ are identified with their projections on the spaces X and X^+ , respectively, hence their elements may be seen as equivalence classes of functions coinciding almost everywhere. In particular, the values of a function in X or X^+ at the endpoints of the domain interval are not relevant to that function being an element of X_C or X_C^+ , respectively. The same is true for the endpoints of domain pieces for elements of X_C if $h < \tau$. $\triangleleft$

In the following sections, some hypotheses on the discretization nodes in $[0, h]$ and on $\mathcal{F}_s$ and V are needed beyond the assumption of [Corollaries 5.2](#) and [5.7](#), in order to attain the regularity required to ensure the convergence of the method. They are all referenced individually from the following list where needed:

(H5.1) the meshes $\{\Omega_N^+\}_{N>0}$ are the Chebyshev zeros

$$t_{N,n} := \frac{h}{2} \left(1 - \cos \left(\frac{(2n-1)\pi}{2N} \right) \right), \quad n \in \{1, \dots, N\},$$

(see [Remark 5.18](#));

(H5.2) the hypothesis of [Corollary 5.2](#) holds;

(H5.3) $\mathcal{F}_s V^+ : X^+ \rightarrow X^+$ has range contained in X_C^+ and $\mathcal{F}_s V^+ : X^+ \rightarrow X_C^+$ is bounded;

(H5.4) $\mathcal{F}_s V^- : X \rightarrow X^+$ has range contained in X_C^+ and $\mathcal{F}_s V^- : X \rightarrow X_C^+$ is bounded.

With respect to [\(5.9\)](#) and [\(5.11\)](#), hypotheses (H5.3) and (H5.4) are fulfilled if the kernel C of [\(5.2\)](#) satisfies the assumptions of [Proposition 2.7](#), which implies the first part of hypotheses (H5.3) and (H5.4), with boundedness following immediately. Eventually, observe that the first condition of [Proposition 2.7](#), namely that $C(t, \theta)$ is essentially bounded on $K \times [a, b]$ for each compact set $K \subset \mathbb{R}$, implies also hypothesis (H5.2). Indeed, the interval $[0, \tau]$ can be partitioned into finitely many subintervals $J_1, \dots, J_n$, each of length less than $\frac{1}{M_{[0, \tau]}}$, such that, for any $s \in \mathbb{R}$ and all $i \in \{1, \dots, n\}$,

$$\operatorname{ess\,sup}_{\sigma \in J_i} \int_{J_i} |C(s+t, \sigma-t)| \, dt \leq M_{[0, \tau]} \int_{J_i} dt < 1.$$

Anyway, in the sequel we base the proofs on hypotheses (H5.2), (H5.3) and (H5.4) in the case one uses operators V and $\mathcal{F}_s$ more general than or different from [\(5.9\)](#) and [\(5.11\)](#).

5.4.2 Well-posedness of the collocation equation

With reference to [\(5.14\)](#), let $\varphi \in X$ and consider the collocation equation

$$W = R_N^+ \mathcal{F}_s V(\varphi, P_N^+ W) \quad (5.16)$$

in $W \in X_N^+$. Observe that we need to assume that $\mathcal{F}_s V: X \times X^+ \rightarrow X^+$ has range in $\tilde{X}^+$: hypotheses (H5.3) and (H5.4) imply that such a subspace $\tilde{X}^+ \subset X^+$ exists.

The aim of this section is to show that (5.16) has a unique solution and to study its relation to the unique solution $w^* \in X^+$ of (5.13). Using (5.10), the equations (5.13) and (5.16) can be rewritten, respectively, as

$$(I_{X^+} - \mathcal{F}_s V^+)w = \mathcal{F}_s V^- \varphi$$

and

$$(I_{X_N^+} - R_N^+ \mathcal{F}_s V^+ P_N^+)W = R_N^+ \mathcal{F}_s V^- \varphi. \quad (5.17)$$

The following preliminary result concerns the operators

$$I_{\tilde{X}^+} - \mathcal{L}_N^+ \mathcal{F}_s V^+|_{\tilde{X}^+}: \tilde{X}^+ \rightarrow \tilde{X}^+, \quad (5.18)$$

and

$$I_{X_N^+} - R_N^+ \mathcal{F}_s V^+ P_N^+: X_N^+ \rightarrow X_N^+. \quad (5.19)$$

Proposition 5.9. *If the operator (5.18) is invertible on $\tilde{X}^+$, then the operator (5.19) is invertible on X_N^+ . Moreover, given $\bar{W} \in X_N^+$, the unique solution $\hat{w} \in \tilde{X}^+$ of*

$$(I_{\tilde{X}^+} - \mathcal{L}_N^+ \mathcal{F}_s V^+|_{\tilde{X}^+})w = P_N^+ \bar{W} \quad (5.20)$$

and the unique solution $\hat{W} \in X_N^+$ of

$$(I_{X_N^+} - R_N^+ \mathcal{F}_s V^+ P_N^+)W = \bar{W} \quad (5.21)$$

are related by $\hat{W} = R_N^+ \hat{w}$ and $\hat{w} = P_N^+ \hat{W}$.

Proof. Apply Proposition 2.18 with $U := \tilde{X}^+$, $V := X_N^+$, $A := \mathcal{F}_s V^+|_{\tilde{X}^+}$, $P := P_N^+$, $R := R_N^+$, recalling (4.13). $\square$

As observed above, the equation (5.16) is equivalent to (5.17), hence, by choosing

$$\bar{W} = R_N^+ \mathcal{F}_s V^- \varphi, \quad (5.22)$$

it is equivalent to (5.21). Observe also that thanks to (4.13) the equation

$$w = \mathcal{L}_N^+ \mathcal{F}_s V(\varphi, w) \quad (5.23)$$

can be rewritten as

$$(I_{X^+} - \mathcal{L}_N^+ \mathcal{F}_s V^+)w = \mathcal{L}_N^+ \mathcal{F}_s V^- \varphi = P_N^+ R_N^+ \mathcal{F}_s V^- \varphi,$$

which is equivalent to (5.20) with the choice (5.22). Thus, by Proposition 5.9, if the operator (5.18) is invertible, then the equation (5.16) has a unique solution $W^* \in X_N^+$ such that

$$W^* = R_N^+ w_N^*, \quad w_N^* = P_N^+ W^*, \quad (5.24)$$

where $w_N^* \in X^+$ is the unique solution of (5.23). Note for clarity that (5.22) implies $w_N^* = \hat{w}$ for $\hat{w}$ in Proposition 5.9.

So, now we show that under suitable assumptions the operator

$$I_{X^+} - \mathcal{L}_N^+ \mathcal{F}_s V^+ : X^+ \rightarrow X^+ \quad (5.25)$$

is invertible, hence proving the invertibility of (5.18). Indeed, since (5.18) is surjective and it is reasonable to assume that

$$(I_{X^+} - \mathcal{L}_N^+ \mathcal{F}_s V^+)(\tilde{X}^+) \subset \tilde{X}^+,$$

as already observed in section 5.3 and thanks to hypothesis (H5.3), if (5.25) is invertible then also (5.18) is.

The next proof follows that of Proposition 4.5, but it uses a different result to ensure the norm convergence of $\mathcal{L}_N^+$ to I_{X^+} , due to the diverse function spaces employed in this case.

Proposition 5.10. *If hypotheses (H5.1), (H5.2) and (H5.3) hold, then there exists a positive integer N_0 such that, for any $N \geq N_0$, the operator*

$$I_{X^+} - \mathcal{L}_N^+ \mathcal{F}_s V^+ : X^+ \rightarrow X^+$$

is invertible and

$$\|(I_{X^+} - \mathcal{L}_N^+ \mathcal{F}_s V^+)^{-1}\|_{X^+ \leftarrow X^+} \leq 2\|(I_{X^+} - \mathcal{F}_s V^+)^{-1}\|_{X^+ \leftarrow X^+}.$$

Moreover, for each $\varphi \in X$, (5.23) has a unique solution $w_N^* \in X^+$ and

$$\|w_N^* - w^*\|_{X^+} \leq 2\|(I_{X^+} - \mathcal{F}_s V^+)^{-1}\|_{X^+ \leftarrow X^+} \|\mathcal{L}_N^+ w^* - w^*\|_{X^+},$$

where $w^* \in X^+$ is the unique solution of (5.13).

Proof. In this proof, let $I := I_{X^+}$. By Theorem 2.16, assuming hypothesis (H5.1), if $w \in X_C^+$, then

$$\|(\mathcal{L}_N^+ - I)w\|_{X^+} \xrightarrow{N \rightarrow +\infty} 0.$$

By the Banach–Steinhaus theorem (Theorem 2.20), the sequence $\|(\mathcal{L}_N^+ - I)|_{X_C^+}\|_{X^+ \leftarrow X_C^+}$ is bounded, hence

$$\|(\mathcal{L}_N^+ - I)|_{X_C^+}\|_{X^+ \leftarrow X_C^+} \xrightarrow{N \rightarrow +\infty} 0. \quad (5.26)$$

Assuming hypothesis (H5.3), this implies

$$\|(\mathcal{L}_N^+ - I)\mathcal{F}_s V^+\|_{X^+ \leftarrow X^+} \leq \|(\mathcal{L}_N^+ - I)|_{X_C^+}\|_{X^+ \leftarrow X_C^+} \|\mathcal{F}_s V^+\|_{X_C^+ \leftarrow X^+} \xrightarrow{N \rightarrow +\infty} 0. \quad (5.27)$$

There exists in particular a positive integer N_0 such that, for each integer $N \geq N_0$,

$$\|(\mathcal{L}_N^+ - I)\mathcal{F}_s V^+\|_{X^+ \leftarrow X^+} \leq \frac{1}{2\|(I - \mathcal{F}_s V^+)^{-1}\|_{X^+ \leftarrow X^+}},$$

i.e.,

$$\|(\mathcal{L}_N^+ - I)\mathcal{F}_s V^+\|_{X^+ \leftarrow X^+} \|(I - \mathcal{F}_s V^+)^{-1}\|_{X^+ \leftarrow X^+} \leq \frac{1}{2}$$

(recall that $I - \mathcal{F}_s V^+$ is invertible with bounded inverse by virtue of [hypothesis \(H5.2\)](#) and [Corollary 5.2](#)). Considering the operator $I - \mathcal{L}_N^+ \mathcal{F}_s V^+$ as a perturbed version of $I - \mathcal{F}_s V^+$ and writing

$$I - \mathcal{L}_N^+ \mathcal{F}_s V^+ = I - \mathcal{F}_s V^+ - (\mathcal{L}_N^+ - I) \mathcal{F}_s V^+,$$

by [Theorem 2.21](#) for each $N \geq N_0$ the operator $I - \mathcal{L}_N^+ \mathcal{F}_s V^+$ is invertible and

$$\begin{aligned} \|(I - \mathcal{L}_N^+ \mathcal{F}_s V^+)^{-1}\|_{X^+ \leftarrow X^+} &\leq \frac{\|(I - \mathcal{F}_s V^+)^{-1}\|_{X^+ \leftarrow X^+}}{1 - \|(I - \mathcal{F}_s V^+)^{-1}((\mathcal{L}_N^+ - I) \mathcal{F}_s V^+)\|_{X^+ \leftarrow X^+}} \\ &\leq 2\|(I - \mathcal{F}_s V^+)^{-1}\|_{X^+ \leftarrow X^+}. \end{aligned}$$

Hence, for fixed $\varphi \in X$, [\(5.23\)](#) has a unique solution $w_N^* \in X^+$. For the same φ , let $e_N^* \in X^+$ such that $w_N^* = w^* + e_N^*$, where $w^* \in X^+$ is the unique solution of [\(5.13\)](#). Then

$$\begin{aligned} w^* + e_N^* &= \mathcal{L}_N^+ \mathcal{F}_s V(\varphi, w^* + e_N^*) \\ &= \mathcal{L}_N^+ \mathcal{F}_s V(\varphi, w^*) + \mathcal{L}_N^+ \mathcal{F}_s V^+ e_N^* \\ &= \mathcal{L}_N^+ w^* + \mathcal{L}_N^+ \mathcal{F}_s V^+ e_N^* \end{aligned}$$

and

$$(I - \mathcal{L}_N^+ \mathcal{F}_s V^+) e_N^* = (\mathcal{L}_N^+ - I) w^*,$$

completing the proof. $\square$

5.4.3 Convergence of the eigenvalues

As in [subsection 4.4.3](#), the operators $T_{M,N}$ and T cannot be compared directly, since they are defined on different spaces. The line of this subsection is the same as in the RFDE case: we successively replace the finite-dimensional operator $T_{M,N}$ with finite-rank operators $\hat{T}_{M,N}$ and $\hat{T}_N$ defined on X ([Propositions 5.11](#) and [5.12](#)) and then prove in [Proposition 5.14](#) that $\hat{T}_N$ converges in operator norm to T . By applying again results from spectral approximation theory [[27](#)] ([Lemma 2.25](#)), we obtain the desired convergence of the eigenvalues of $T_{M,N}$ to the eigenvalues of T ([Proposition 5.15](#) and [Theorem 5.16](#)), along with similar estimates of the convergence order, depending on the smoothness of the eigenfunctions of T (i.e., presumably on the regularity of the model coefficients, see [chapter 9](#)).

Unlike in [Proposition 4.8](#), there is no need to restrict the operators to subspaces of X . Indeed, this follows from [hypothesis \(H5.4\)](#) (compare it with [hypothesis \(H4.4\)](#)), which ensures that not only $\mathcal{F}_s V^+$, but also $\mathcal{F}_s V^-$ has a regularization effect on its arguments. This is a major difference between the RFDE and the RE cases and depends on the form of the different types of equations. [Chapter 7](#) contains a detailed discussion of the interplay between the types of equations, the regularizing effects and the structure of the proofs, including the consequences of different choices in function subspaces to restrict the operators.

The first step in the convergence proof is to introduce the finite-rank operator $\hat{T}_{M,N}$ on X and show the relation between its spectrum and that of $T_{M,N}$. Notice that $\hat{T}_{M,N}$ must be defined on a Banach space contained in $\tilde{X}$, due to the use of R_M , hence the natural choice of X_C .

Proposition 5.11. *The finite-dimensional operator $T_{M,N}$ has the same nonzero eigenvalues, with the same geometric and partial multiplicities, of the operator*

$$\hat{T}_{M,N} := P_M T_{M,N} R_M|_{X_C} : X_C \rightarrow X_C.$$

Moreover, if $\Phi \in X_M$ is an eigenvector of $T_{M,N}$ associated with a nonzero eigenvalue μ , then $P_M \Phi \in X_C$ is an eigenvector of $\hat{T}_{M,N}$ associated with the same eigenvalue μ .

Proof. Apply [Proposition 2.22](#) with $U := X_C$, $V := X_M$, $A := T_{M,N}$, $P := P_M$, $R := R_M$, recalling [\(4.11\)](#), since prolongations are polynomials, hence continuous. $\square$

Define the operator $\hat{T}_N : X \rightarrow X$ as

$$\hat{T}_N \varphi := V(\varphi, w_N^*)_h,$$

where $w_N^* \in X^+$ is the solution of the fixed point equation [\(5.23\)](#), which, under [hypotheses \(H5.1\), \(H5.2\) and \(H5.3\)](#), is unique thanks to [Propositions 5.9 and 5.10](#). Observe that w_N^* is a polynomial. Then, for $\varphi \in X_C$, by [\(5.24\)](#),

$$\begin{aligned} \hat{T}_{M,N} \varphi &= P_M T_{M,N} R_M \varphi \\ &= P_M R_M V(P_M R_M \varphi, P_N^+ W^*)_h \\ &= \mathcal{L}_M V(\mathcal{L}_M \varphi, w_N^*)_h \\ &= \mathcal{L}_M \hat{T}_N \mathcal{L}_M \varphi, \end{aligned}$$

where $W^* \in X_N^+$ and $w_N^* \in X^+$ are the solutions, respectively, of [\(5.14\)](#) applied to $\Phi = R_M \varphi$ and of [\(5.23\)](#) with $\mathcal{L}_M \varphi$ replacing φ . These solutions are unique under [hypotheses \(H5.1\), \(H5.2\) and \(H5.3\)](#), thanks again to [Propositions 5.9 and 5.10](#).

Now we show the relation between the spectra of $\hat{T}_{M,N}$ and $\hat{T}_N$. The proof follows that of [Proposition 4.7](#).

Proposition 5.12. *Assume that [hypotheses \(H5.1\), \(H5.2\) and \(H5.3\)](#) hold and let $M \geq N \geq N_0$, with N_0 given by [Proposition 5.10](#). Then the operator $\hat{T}_{M,N}$ has the same nonzero eigenvalues, with the same geometric and partial multiplicities and associated eigenvectors, of the operator $\hat{T}_N$.*

Proof. Denote by Π_r and Π_r^+ the subspaces of polynomials of degree r of X and X^+ , respectively, and observe that [Remark 5.8](#) applies also here. Note that $w_N^* \in \Pi_{N-1}^+$.

If $h \geq \tau$, for all $\varphi \in X$, $\hat{T}_N \varphi = V(\varphi, w_N^*)_h \in \Pi_{N-1}$. Thus both $\hat{T}_N$ and $\hat{T}_{M,N} = \mathcal{L}_M \hat{T}_N \mathcal{L}_M$ have range contained in Π_M , being $M \geq N$. By [Proposition 2.23](#) and [Remark 2.24](#), $\hat{T}_N$ and $\hat{T}_{M,N}$ have the same nonzero eigenvalues, with the same geometric and partial multiplicities and associated eigenvectors, as their restrictions to Π_M . Observing that

$$\hat{T}_{M,N}|_{\Pi_M} = \mathcal{L}_M \hat{T}_N \mathcal{L}_M|_{\Pi_M} = \hat{T}_N|_{\Pi_M},$$

the thesis follows.

Consider now the case $h < \tau$. Denote by Π_r^{pw} the subspace of piecewise polynomials of degree r of X on the intervals $[\theta^{(q+1)}, \theta^{(q)}]$, for $q = 0, \dots, Q-1$. For all $\varphi \in \Pi_M^{\text{pw}}$, $\hat{T}_N \varphi = V(\varphi, w_N^*)_h \in \Pi_{N-1}^{\text{pw}} \subset \Pi_M^{\text{pw}}$. Let $\mu \neq 0$, $\varphi \in X$ and $\bar{\varphi} \in \Pi_M^{\text{pw}}$ such that

$$(\mu I_X - \hat{T}_N) \varphi = \mu \varphi - V(\varphi, w_N^*)_h = \bar{\varphi}.$$

This equation can be rewritten as

$$\begin{cases} \mu \varphi(\theta) = w_N^*(h + \theta) + \bar{\varphi}(\theta), & \text{if } \theta \in (-h, 0], \\ \mu \varphi(\theta) = \varphi(h + \theta) + \bar{\varphi}(\theta), & \text{if } \theta \in [-\tau, -h]. \end{cases}$$

From the first equation, φ restricted to $[-h, 0]$ is a polynomial of degree M . From the second equation it is easy to show that $\varphi \in \Pi_M^{\text{pw}}$ by induction on the intervals $[\theta^{(q+1)}, \theta^{(q)}]$, for $q = 1, \dots, Q-1$. Hence, by [Proposition 2.23](#), $\hat{T}_N$ has the same nonzero eigenvalues, with the same geometric and partial multiplicities and associated eigenvectors, as its restriction to Π_M^{pw} . The same holds for $\hat{T}_{M,N} = \mathcal{L}_M \hat{T}_N \mathcal{L}_M$ by [Proposition 2.23](#) and [Remark 2.24](#) since its range is contained in Π_M^{pw} . The thesis follows by observing that

$$\hat{T}_{M,N} \upharpoonright_{\Pi_M^{\text{pw}}} = \mathcal{L}_M \hat{T}_N \mathcal{L}_M \upharpoonright_{\Pi_M^{\text{pw}}} = \hat{T}_N \upharpoonright_{\Pi_M^{\text{pw}}}. \quad \square$$

Below we prove the norm convergence of $\hat{T}_N$ to T , which is the key step to obtain the main result of this chapter. Similarly to [Lemma 4.9](#), we first need to extend the results of [Corollary 5.7](#) to $(I_{X^+} - \mathcal{F}_s V^+) \upharpoonright_{X_C^+}$ in the following lemma.

Lemma 5.13. *If [hypotheses \(H5.2\)](#) and [\(H5.3\)](#) hold, then $(I_{X^+} - \mathcal{F}_s V^+) \upharpoonright_{X_C^+}$ is invertible with bounded inverse.*

Proof. Since $I_{X^+} - \mathcal{F}_s V^+$ is invertible with bounded inverse by virtue of [hypothesis \(H5.2\)](#) and [Corollary 5.7](#), given $f \in X_C^+$ the equation $(I_{X^+} - \mathcal{F}_s V^+)w = f$ has a unique solution $w \in X^+$, which by [hypothesis \(H5.3\)](#) is in X_C^+ . Hence, the operator $(I_{X^+} - \mathcal{F}_s V^+) \upharpoonright_{X_C^+}$ is invertible. It is also bounded, since $\|\cdot\|_{X^+} \leq h \|\cdot\|_{X_C^+}$ in X_C^+ , which implies

$$\|\mathcal{F}_s V^+ \upharpoonright_{X_C^+}\|_{X_C^+ \leftarrow X_C^+} \leq h \|\mathcal{F}_s V^+ \upharpoonright_{X_C^+ \leftarrow X^+}.$$

The bounded inverse theorem ([Theorem 2.19](#)) completes the proof. $\square$

Proposition 5.14. *If [hypotheses \(H5.1\)](#), [\(H5.2\)](#), [\(H5.3\)](#) and [\(H5.4\)](#) hold, then*

$$\|\hat{T}_N - T\|_{X \leftarrow X} \xrightarrow{N \rightarrow +\infty} 0.$$

Proof. Let $\varphi \in X$ and let w^* and w_N^* be the solutions of the fixed point equations [\(5.13\)](#) and [\(5.23\)](#), respectively. Then

$$(\hat{T}_N - T)\varphi = V(\varphi, w_N^*)_h - V(\varphi, w^*)_h = V^+(w_N^* - w^*)_h.$$

Assuming [hypotheses \(H5.3\)](#) and [\(H5.4\)](#) and recalling that

$$w^* = \mathcal{F}_s V^+ w^* + \mathcal{F}_s V^- \varphi,$$

it is clear that $w^* \in X_C^+$. Assuming also [hypotheses \(H5.1\)](#) and [\(H5.2\)](#), by [Proposition 5.10](#), there exists a positive integer N_0 such that, for any $N \geq N_0$,

$$\begin{aligned} \|(\hat{T}_N - T)\varphi\|_X &= \|V^+(w_N^* - w^*)_h\|_X \\ &\leq \|w_N^* - w^*\|_{X^+} \\ &\leq 2\|(I_{X^+} - \mathcal{F}_s V^+)^{-1}\|_{X^+ \leftarrow X^+} \|\mathcal{L}_N^+ w^* - w^*\|_{X^+} \quad (5.28) \\ &\leq 2\|(I_{X^+} - \mathcal{F}_s V^+)^{-1}\|_{X^+ \leftarrow X^+} \\ &\quad \|(\mathcal{L}_N^+ - I_{X^+})\upharpoonright_{X_C^+}\|_{X^+ \leftarrow X_C^+} \|w^*\|_{X_C^+} \end{aligned}$$

holds by virtue of [\(5.26\)](#). Eventually,

$$\|w^*\|_{X_C^+} \leq \|((I_{X^+} - \mathcal{F}_s V^+)^{-1})\upharpoonright_{X_C^+}\|_{X_C^+ \leftarrow X_C^+} \|\mathcal{F}_s V^-\|_{X_C^+ \leftarrow X} \|\varphi\|_X \quad (5.29)$$

completes the proof thanks to [Lemma 5.13](#) and [hypothesis \(H5.4\)](#). $\square$

With respect to the analogous [Proposition 4.10](#), [Proposition 5.14](#) turns out to be stronger, stating the norm convergence of the operators on their natural domain, instead of some proper subspace.

As in [chapter 4](#), the final convergence result is obtained thanks to results from spectral approximation theory, namely the ones summarized in [Lemma 2.25](#), and classic results in interpolation theory.

Proposition 5.15. *Assume that [hypotheses \(H5.1\)](#), [\(H5.2\)](#), [\(H5.3\)](#) and [\(H5.4\)](#) hold. If $\mu \in \mathbb{C} \setminus \{0\}$ is an eigenvalue of T with finite algebraic multiplicity ν and ascent l , and Δ is a neighborhood of μ such that μ is the only eigenvalue of T in Δ , then there exists a positive integer $N_1 \geq N_0$, with N_0 given by [Proposition 5.10](#), such that, for any $N \geq N_1$, $\hat{T}_N$ has in Δ exactly ν eigenvalues $\mu_{N,j}$, $j \in \{1, \dots, \nu\}$, counting their multiplicities. Moreover, if for each $\varphi \in \mathcal{E}_\mu$, where $\mathcal{E}_\mu$ is the generalized eigenspace of T associated with μ , the function w^* that solves [\(5.13\)](#) is of class C^p , with $p \geq 1$, then*

$$\max_{j \in \{1, \dots, \nu\}} |\mu_{N,j} - \mu| = o(N^{\frac{1-p}{l}}).$$

Proof. By [Proposition 5.14](#),

$$\|\hat{T}_N - T\|_{X \leftarrow X} \xrightarrow{N \rightarrow +\infty} 0.$$

The first part of the thesis is obtained by applying [Lemma 2.25](#). From the same [Lemma 2.25](#), [\(2.16\)](#) follows with

$$\epsilon_N := \|(\hat{T}_N - T)\upharpoonright_{\mathcal{E}_\mu}\|_{X \leftarrow \mathcal{E}_\mu}$$

and $\mathcal{E}_\mu$ the generalized eigenspace of μ equipped with the norm of X restricted to $\mathcal{E}_\mu$.

Let $\varphi_1, \dots, \varphi_\nu$ be a basis of $\mathcal{E}_\mu$. An element φ of $\mathcal{E}_\mu$ can be written as

$$\varphi = \sum_{j=1}^{\nu} \alpha_j(\varphi) \varphi_j,$$

with $\alpha_j(\varphi) \in \mathbb{C}$, for $j \in \{1, \dots, \nu\}$, hence

$$\|(\hat{T}_N - T)\varphi\|_X \leq \max_{j \in \{1, \dots, \nu\}} |\alpha_j(\varphi)| \sum_{j=1}^{\nu} \|(\hat{T}_N - T)\varphi_j\|_X.$$

The functional

$$\varphi \mapsto \max_{j \in \{1, \dots, v\}} |\alpha_j(\varphi)|$$

is a norm on $\mathcal{E}_\mu$, so it is equivalent to the norm of X restricted to $\mathcal{E}_\mu$. Thus, there exists a positive constant c independent of φ such that

$$\max_{j \in \{1, \dots, v\}} |\alpha_j(\varphi)| \leq c \|\varphi\|_X$$

and

$$\epsilon_N = \|(\hat{T}_N - T)|_{\mathcal{E}_\mu}\|_{X \leftarrow \mathcal{E}_\mu} \leq c \sum_{j=1}^v \|(\hat{T}_N - T)\varphi_j\|_X.$$

Let $j \in \{1, \dots, v\}$. As seen in [Proposition 5.14](#),

$$\|(\hat{T}_N - T)\varphi_j\|_X \leq 2\|(I - \mathcal{F}_s V^+)^{-1}\|_{X^+ \leftarrow X^+} \|(\mathcal{L}_N^+ - I)w_j^*\|_{X^+},$$

where w_j^* is the solution of (5.13) associated with φ_j . Now, by [Theorem 2.14](#) and [Corollary 2.12](#), since w_j^* is of class C^p , the bound

$$\begin{aligned} \|(\mathcal{L}_N^+ - I)w_j^*\|_{X^+} &\leq h(1 + \Lambda_N)E_{N-1}(w_j^*) \\ &\leq h(1 + \Lambda_N) \frac{6^{p+1}e^p}{1+p} \left(\frac{h}{2}\right)^p \frac{1}{(N-1)^p} \omega\left(\frac{h}{2(N-1-p)}\right) \end{aligned}$$

holds, where Λ_N is the Lebesgue constant for Ω_N^+ , $E_{N-1}(\cdot)$ is the best uniform approximation error and $\omega(\cdot)$ is the modulus of continuity of $(w_j^*)^{(p)}$ on $[0, h]$. Since [hypothesis \(H5.1\)](#) is assumed, by [Theorem 2.15](#), $\Lambda_N = o(N)$. Hence, $\epsilon_N = o(N^{1-p})$ and the thesis follows immediately. $\square$

Theorem 5.16. *Assume that [hypotheses \(H5.1\)](#), [\(H5.2\)](#), [\(H5.3\)](#) and [\(H5.4\)](#) hold. If $\mu \in \mathbb{C} \setminus \{0\}$ is an eigenvalue of T with finite algebraic multiplicity v and ascent l , and Δ is a neighborhood of μ such that μ is the only eigenvalue of T in Δ , then there exists a positive integer $N_1 \geq N_0$, with N_0 given by [Proposition 5.10](#), such that, for any $N \geq N_1$ and any $M \geq N$, $T_{M,N}$ has in Δ exactly v eigenvalues $\mu_{M,N,j}$, $j \in \{1, \dots, v\}$, counting their multiplicities. Moreover, if for each $\varphi \in \mathcal{E}_\mu$, where $\mathcal{E}_\mu$ is the generalized eigenspace of T associated with μ , the function w^* that solves (5.13) is of class C^p , with $p \geq 1$, then*

$$\max_{j \in \{1, \dots, v\}} |\mu_{M,N,j} - \mu| = o(N^{\frac{1-p}{1-l}}).$$

Proof. If $M \geq N \geq N_0$, by [Propositions 5.11](#) and [5.12](#) the operators $T_{M,N}$, $\hat{T}_{M,N}$ and $\hat{T}_N$ have the same nonzero eigenvalues, with the same geometric and partial multiplicities and associated eigenvectors. The thesis follows by [Proposition 5.15](#). $\square$

The comments of [Remark 4.15](#) on the further error contribution due to quadrature of the integrals in $\mathcal{F}_s$ apply also to this case.

Remark 5.17. For REs, it is not clear whether it is possible to satisfy the additional hypothesis of [Theorem 5.16](#) implying the result on the convergence order. Indeed, the argumentation of [Remark 4.13](#) does not hold in this case, since REs describe the values of the function itself and the regularizing effect of the equation is not present. Nevertheless we can expect the regularity of

solutions to be related to properties of the integration kernel $C(t, \theta)$. Furthermore, the examples of [chapter 8](#) show that in practice the convergence happens with infinite order. See also [chapter 9](#). $\triangleleft$

Remark 5.18. Nodes other than those required by [hypothesis \(H5.1\)](#) may be used. Indeed, they are only asked to satisfy the hypotheses of [Theorem 2.16](#) and $\Lambda_N = o(N)$. Let us notice that both are guaranteed by zeros of other families of classic orthogonal polynomials [[22](#)]. Anyway, here we assume [hypothesis \(H5.1\)](#) since these are the nodes we actually use in implementing the method (see [section A.2](#) for more details on implementation choices). $\triangleleft$

In this chapter we extend the method of [15, 16] presented in chapters 4 and 5 to coupled renewal equations and retarded functional differential equations (coupled REs/RFDEs or coupled equations).

Many of the comments in chapter 5 apply also here and the proof of convergence turns out to be a combination of the RFDE and RE cases. However, while developing the method for coupled equations, the convergence proof had become in several places more involved (and at some point it even seemed unattainable!), due to a different choice of the subspaces of the state space for restricting the operators. Chapter 7 contains some comments on this, explaining how the structure of the proof is affected by the regularization properties of $\mathcal{F}_s$, which vary between the different types of equation.

6.1 EVOLUTION OPERATORS FOR LINEAR COUPLED REs/RFDEs

Let $d_X, d_Y \in \mathbb{N}$ with $d_X + d_Y > 0$ and $\tau \in \mathbb{R}$ positive and consider the function spaces $X := L^1([-\tau, 0], \mathbb{R}^{d_X})$ and $Y := C([-\tau, 0], \mathbb{R}^{d_Y})$ equipped with the usual L^1 (5.1) and uniform (4.1) norm, respectively. Consider also the function space $X \times Y$, equipped with the norm defined as*

$$\|(x, y)\|_{X \times Y} := \|x\|_X + \|y\|_Y. \quad (6.1)$$

A linear coupled RE/RFDE with finite delay is a relation of the form

$$\begin{cases} x(t) = \int_{-\tau}^0 C_{XX}(t, \theta) x_t(\theta) d\theta + L_{XY}(t) y_t, \\ y'(t) = \int_{-\tau}^0 C_{YX}(t, \theta) x_t(\theta) d\theta + L_{YY}(t) y_t, \end{cases} \quad t \in \mathbb{R}, \quad (6.2)$$

where y' denotes the right-hand derivative of y , x_t and y_t are defined as in (2.1), $C_{XX}: \mathbb{R} \times [-\tau, 0] \rightarrow \mathbb{R}^{d_X \times d_X}$ and $C_{YX}: \mathbb{R} \times [-\tau, 0] \rightarrow \mathbb{R}^{d_Y \times d_X}$ are measurable functions and $\mathbb{R} \times Y \ni (t, \psi) \mapsto L_{XY}(t)\psi \in \mathbb{R}^{d_X}$ and $\mathbb{R} \times Y \ni (t, \psi) \mapsto L_{YY}(t)\psi \in \mathbb{R}^{d_Y}$ are continuous functions, linear in the second argument. This condition implies that $L_{XY}(t): Y \rightarrow \mathbb{R}^{d_X}$ and $L_{YY}(t): Y \rightarrow \mathbb{R}^{d_Y}$ are linear bounded functionals for all $t \in \mathbb{R}$ and $L_{XY}(\cdot)\psi: \mathbb{R} \rightarrow \mathbb{R}^{d_X}$ and $L_{YY}(\cdot)\psi: \mathbb{R} \rightarrow \mathbb{R}^{d_Y}$ are continuous functions for all $\psi \in Y$. The numbers $d_X + d_Y$ and τ are, respectively, the dimension of the equation and the maximum delay, while $X \times Y$ is the state space and $(x_t, y_t) \in X \times Y$ is the state at time t .

* Different choices are possible for the norm of a product space: as an example, the norm defined as $\max\{\|x\|_X, \|y\|_Y\}$ is equivalent to (6.1).

For $s \in \mathbb{R}$ and $(\varphi, \psi) \in X \times Y$, the Cauchy problem for (6.2) is defined as

$$\begin{cases} x(t) = \int_{-\tau}^0 C_{XX}(t, \theta) x_t(\theta) d\theta + L_{XY}(t) y_t, & t > s, \\ y'(t) = \int_{-\tau}^0 C_{YX}(t, \theta) x_t(\theta) d\theta + L_{YY}(t) y_t, & t \geq s, \\ x_s = \varphi, \\ y_s = \psi. \end{cases} \quad (6.3)$$

A pair of functions (x, y) is a solution of (6.3) on $[s - \tau, s + t_f]$ if there exists $t_f > 0$ such that $x \in L_{\text{loc}}^1([s - \tau, s + t_f], \mathbb{R}^{d_X})$, $y \in C([s - \tau, s + t_f], \mathbb{R}^{d_Y})$, $x_s = \varphi$, $y_s = \psi$ and for each $t \in [s, s + t_f]$ $x(t)$ and $y(t)$ satisfy (6.2). The final time t_f may be $+\infty$. To emphasize the dependence of solutions on both the initial time s and the initial functions (φ, ψ) , a solution $(x(\cdot), y(\cdot))$ of (6.3) is sometimes denoted as $(x(\cdot; s, \varphi), y(\cdot; s, \psi))$.

Observe that the difference in the treatment of the initial time s between RFDEs and REs is maintained in the coupled case between the renewal and the differential parts of the equation (compare (6.3) with (4.3) and (5.3)). Remark 5.6 remains valid also in this case.

Theorem 6.1. *If*

- $(t, \theta) \mapsto |C_{XX}(t, \theta)|$ and $(t, \theta) \mapsto |C_{YX}(t, \theta)|$ are essentially bounded on $\mathbb{R} \times [-\tau, 0]$,
- $t \mapsto \|L_{XY}(t)\|_{\mathbb{R}^{d_X} \leftarrow Y}$ and $t \mapsto \|L_{YY}(t)\|_{\mathbb{R}^{d_Y} \leftarrow Y}$ are bounded on $\mathbb{R}$,
- the function $t \mapsto C_{YX}(t, \theta)$ is continuous for almost all $\theta \in [-\tau, 0]$, uniformly with respect to θ ,

then the Cauchy problem

$$\begin{cases} x(t) = \int_{-\tau}^0 C_{XX}(s + t, \theta) x_t(\theta) d\theta + L_{XY}(s + t) y_t, & t > 0, \\ y'(t) = \int_{-\tau}^0 C_{YX}(s + t, \theta) x_t(\theta) d\theta + L_{YY}(s + t) y_t, & t \geq 0, \\ x_0 = \phi \in X, \\ y_0 = \psi \in Y, \end{cases} \quad (6.4)$$

has a unique solution $(x(\cdot; s, \varphi), y(\cdot; s, \psi))$ on $[-\tau, +\infty)$.

We postpone the proof of Theorem 6.1 to section 6.2 (page 83) in order to take advantage of the notations defined therein.

Theorem 6.1 ensures the uniqueness of solutions of (6.3), equivalent to (6.4), under suitable conditions, allowing us to define the family of evolution operators $\{T(t, s)\}_{(t, s) \in \Delta}$

$$T(t, s): X \times Y \rightarrow X \times Y, \quad T(t, s)(\varphi, \psi) := (x_t(\cdot; s, \varphi), y_t(\cdot; s, \psi)), \quad (6.5)$$

where

$$\Delta := \{(t, s) \in \mathbb{R}^2 \mid -\infty \leq s \leq t \leq +\infty\}.$$

As for the analogous Propositions 4.2 and 5.3, the following proposition is a consequence of Theorem 3.1 and of the correspondence between solutions

of the linear initial value problem and the relevant abstract equation. Again, as for REs, it is also possible to obtain a direct proof as for the analogous [Proposition 4.2](#).

Proposition 6.2. *The family of evolution operators $\{T(t, s)\}_{(t, s) \in \Delta}$ defined in (6.5) is a strongly continuous evolutionary system.*

Let $s \in \mathbb{R}$ and $h \geq 0$ and consider the evolution operator

$$T := T(s + h, s).$$

The aim of this chapter is to approximate the spectrum of T by computing with standard techniques the eigenvalues of a finite-dimensional approximation of T obtained via pseudospectral collocation, as described in [section 6.3](#).

Recall again from [section 1.2](#) and the previous chapters that this can be applied to investigate the stability of equilibria and periodic solutions, and possibly to approximate Lyapunov exponents.

The chosen form for the terms of the right-hand side containing x is the same as the one in [chapter 5](#). Comments similar to those of [Remark 5.4](#) on its generality, also in comparison to (4.15), and on the kinds of equations excluded from our treatment, can be made also in the coupled case.

6.2 REFORMULATION OF T

Define the function spaces

$$X^+ := L^1([0, h], \mathbb{R}^{d_x}), \quad X^\pm := L^1([-\tau, h], \mathbb{R}^{d_x}),$$

equipped with the corresponding L^1 norms denoted, respectively, by $\|\cdot\|_{X^+}$ and $\|\cdot\|_{X^\pm}$, and the function spaces

$$Y^+ := C([0, h], \mathbb{R}^{d_y}), \quad Y^\pm := C([-\tau, h], \mathbb{R}^{d_y}),$$

equipped with the corresponding uniform norms denoted, respectively, by $\|\cdot\|_{Y^+}$ and $\|\cdot\|_{Y^\pm}$. Consider also the function spaces $X^+ \times Y^+$ and $X^\pm \times Y^\pm$, equipped with the norms $\|\cdot\|_{X^+ \times Y^+}$ and $\|\cdot\|_{X^\pm \times Y^\pm}$ defined as in (6.1).

As in the previous chapters, the operator V captures the rule to construct a solution of (6.3), while the operator $\mathcal{F}_s$ applies the right-hand side functional to its argument after a time shift.

The operator V acts on the two spaces $X \times X^+$ and $Y \times Y^+$ as its counterparts for REs and RFDEs, respectively. Indeed, defining the operators $V_X: X \times X^+ \rightarrow X^\pm$ and $V_Y: Y \times Y^+ \rightarrow Y^\pm$ as

$$V_X(\varphi, w)(t) := \begin{cases} w(t), & t \in (0, h], \\ \varphi(t), & t \in [-\tau, 0], \end{cases}$$

$$V_Y(\psi, z)(t) := \begin{cases} \psi(0) + \int_0^t z(\sigma) d\sigma, & t \in (0, h], \\ \psi(t), & t \in [-\tau, 0], \end{cases}$$

the operator $V: (X \times Y) \times (X^+ \times Y^+) \rightarrow X^\pm \times Y^\pm$ is defined as

$$V((\varphi, \psi), (w, z)) := (V_X(\varphi, w), V_Y(\psi, z)).$$

For ease of notation, define also the operators

$$\begin{aligned} V_X^-: X &\rightarrow X^\pm, & V_X^+: X^+ &\rightarrow X^\pm, \\ V_Y^-: Y &\rightarrow Y^\pm, & V_Y^+: Y^+ &\rightarrow Y^\pm, \\ V^-: X \times Y &\rightarrow X^\pm \times Y^\pm, & V^+: X^+ \times Y^+ &\rightarrow X^\pm \times Y^\pm, \end{aligned}$$

respectively, as

$$\begin{aligned} V_X^- \varphi &:= V_X(\varphi, 0_{X^+}), & V_X^+ w &:= V_X(0_X, w), \\ V_Y^- \psi &:= V_Y(\psi, 0_{Y^+}), & V_Y^+ z &:= V_Y(0_Y, z), \\ V^-(\varphi, \psi) &:= V((\varphi, \psi), (0_{X^+}, 0_{Y^+})), & V^+(w, z) &:= V((0_X, 0_Y), (w, z)), \end{aligned}$$

and observe that

$$V_X(\varphi, w) = V_X^- \varphi + V_X^+ w, \quad V_Y(\psi, z) = V_Y^- \psi + V_Y^+ z,$$

and

$$\begin{aligned} V((\varphi, \psi), (w, z)) &= V^-(\varphi, \psi) + V^+(w, z) \\ &= (V_X^- \varphi + V_X^+ w, V_Y^- \psi + V_Y^+ z). \end{aligned} \tag{6.6}$$

As in [chapter 5](#), note that $V_X(\varphi, w)(t)$ can have a discontinuity in 0 even when φ and w are continuous but $\varphi(0) \neq w(0)$.

Define the operators $\mathcal{F}_{X,s}: X^\pm \times Y^\pm \rightarrow X^+ \times Y^+$ and $\mathcal{F}_{Y,s}: X^\pm \times Y^\pm \rightarrow Y^+ \times X^+$ as

$$\begin{aligned} \mathcal{F}_{X,s}(u, v)(t) &:= \int_{-\tau}^0 C_{XX}(s+t, \theta) u(t+\theta) d\theta + L_{XY}(s+t) v_t, \\ \mathcal{F}_{Y,s}(u, v)(t) &:= \int_{-\tau}^0 C_{YX}(s+t, \theta) u(t+\theta) d\theta + L_{YY}(s+t) v_t, \end{aligned}$$

for $t \in [0, h]$, and the operator $\mathcal{F}_s: X^\pm \times Y^\pm \rightarrow X^+ \times Y^+$ as

$$\mathcal{F}_s(u, v) := (\mathcal{F}_{X,s}(u, v), \mathcal{F}_{Y,s}(u, v)).$$

The evolution operator T can be reformulated as

$$T(\varphi, \psi) = V((\varphi, \psi), (w^*, z^*))_h, \tag{6.7}$$

where $(w^*, z^*) \in X^+ \times Y^+$ is the solution of the fixed point equation

$$(w, z) = \mathcal{F}_s V((\varphi, \psi), (w, z)), \tag{6.8}$$

which exists uniquely and bounded thanks to [Corollary 6.4](#) below. Recall that in [\(6.7\)](#) the subscript h is used according to [Definition 2.1](#), hence

$$V((\varphi, \psi), (w^*, z^*))_h(\theta) = V((\varphi, \psi), (w^*, z^*))(h + \theta), \quad \theta \in [-\tau, 0].$$

We now take advantage of the notations defined above for the function spaces and the operators in order to prove [Theorem 6.1](#). To the best of the author's knowledge, this is a novel contribution of this thesis, since a detailed proof of existence and uniqueness for IVPs of coupled REs/RFDEs is absent from the literature. We first need the next proposition.

Proposition 6.3. *If the hypotheses of Proposition 2.7 hold for C_{YX} , then the map*

$$(t, \varphi) \mapsto \int_{-\tau}^0 C_{YX}(t, \theta) \varphi(\theta) d\theta$$

is continuous on $\mathbb{R} \times X$.

Proof. Let $\{(t_n, \varphi_n)\}_{n \in \mathbb{N}}$ be a sequence in $\mathbb{R} \times X$ convergent to $(\bar{t}, \bar{\varphi}) \in \mathbb{R} \times X$. Then

$$\begin{aligned} & \left| \int_{-\tau}^0 C_{YX}(t_n, \theta) \varphi_n(\theta) d\theta - \int_{-\tau}^0 C_{YX}(\bar{t}, \theta) \bar{\varphi}(\theta) d\theta \right| \\ & \leq \int_{-\tau}^0 |C_{YX}(t_n, \theta)| |\varphi_n(\theta) - \bar{\varphi}(\theta)| d\theta \\ & \quad + \int_{-\tau}^0 |C_{YX}(t_n, \theta) - C_{YX}(\bar{t}, \theta)| |\bar{\varphi}(\theta)| d\theta. \end{aligned}$$

By the first hypothesis of Proposition 2.7, choosing a compact $K \subset \mathbb{R}$ containing $\{t_n\}_{n \in \mathbb{N}}$ and $\bar{t}$,

$$\int_{-\tau}^0 |C_{YX}(t_n, \theta)| |\varphi_n(\theta) - \bar{\varphi}(\theta)| d\theta \leq M_K \int_{-\tau}^0 |\varphi_n(\theta) - \bar{\varphi}(\theta)| d\theta \rightarrow 0$$

since $\varphi_n \rightarrow \bar{\varphi}$ in X . By the second hypothesis of Proposition 2.7,

$$|C_{YX}(t_n, \theta) - C_{YX}(\bar{t}, \theta)| \rightarrow 0$$

for almost all θ uniformly in θ , hence, chosen $\bar{\varepsilon} > 0$, for n large enough and almost all θ the vanishing integrand function is dominated by $\bar{\varepsilon} |\bar{\varphi}(\theta)|$, and by Lebesgue's dominated convergence theorem (Theorem 2.29) the second integral converges to 0. $\square$

Proof of Theorem 6.1. By Proposition 6.3, the map

$$(t, \varphi) \mapsto \int_{-\tau}^0 C_{YX}(t, \theta) \varphi(\theta) d\theta$$

is continuous on $\mathbb{R} \times X$ (the assumptions on C_{YX} imply the hypotheses of Proposition 2.7). Since also the map $(t, \psi) \mapsto L_{YX}(t)\psi$ is continuous on $\mathbb{R} \times Y$, the differential equation in (6.3)

$$y'(t) = \int_{-\tau}^0 C_{YX}(s+t, \theta) x(t+\theta) d\theta + L_{YX}(s+t)y_t$$

can be rewritten as

$$y(t) = \psi(0) + \int_0^t \left[\int_{-\tau}^0 C_{YX}(s+\sigma, \theta) x(\sigma+\theta) d\theta + L_{YX}(s+\sigma)y_\sigma \right] d\sigma.$$

Consider the Cauchy problem on $[0, h]$ for some $h > 0$. Define the operator $\mathcal{K}: X^+ \times Y^+ \rightarrow X^+ \times Y^+$ as

$$\begin{aligned} \mathcal{K}(x, y)(t) &= \begin{pmatrix} \mathcal{K}_X(x, y)(t) \\ \mathcal{K}_Y(x, y)(t) \end{pmatrix} \\ &:= \begin{pmatrix} \int_{-\tau}^0 C_{XX}(s+t, \theta) V_X(\varphi, x)(t+\theta) d\theta \\ \quad + L_{XY}(s+t) V_Y(\psi, y)_t \\ \psi(0) + \int_0^t \left[\int_{-\tau}^0 C_{YX}(s+\sigma, \theta) V_X(\varphi, x)(\sigma+\theta) d\theta \right. \\ \quad \left. + L_{YX}(s+\sigma) V_Y(\psi, y)_\sigma \right] d\sigma \end{pmatrix}. \end{aligned}$$

The following inequalities hold:

$$\begin{aligned}
& |\mathcal{K}_Y(x, y)(t) - \mathcal{K}_Y(w, z)(t)| \\
& \leq \int_0^t \int_{-\tau}^0 |C_{YX}(s + \sigma, \theta)| |V_X^+(x - w)(\sigma + \theta)| \, d\theta \, d\sigma \\
& \quad + \int_0^t \|L_{YX}(s + \sigma)\|_{\mathbb{R}^{d_Y \leftarrow Y}} \|V_Y^+(y - z)_\sigma\|_Y \, d\sigma \\
& \leq \bar{C}_{YX} \int_0^t \|V_X^+(x - w)_\sigma\|_X \, d\sigma \\
& \quad + \int_0^t \|L_{YX}(s + \sigma)\|_{\mathbb{R}^{d_Y \leftarrow Y}} \sigma \|y - z\|_{Y^+} \, d\sigma \\
& \leq \bar{C}_{YX} t \|x - w\|_{X^+} + \bar{L}_{YX} \frac{t^2}{2} \|y - z\|_{Y^+},
\end{aligned}$$

where

$$\bar{L}_{YX} := \sup_{t \in \mathbb{R}} \|L_{YX}(t)\|_{\mathbb{R}^{d_Y \leftarrow Y}}$$

and

$$\bar{C}_{YX} := \operatorname{ess\,sup}_{(t, \theta) \in \mathbb{R} \times [-\tau, 0]} |C_{YX}(t, \theta)|.$$

Thus

$$\|\mathcal{K}_Y(x, y) - \mathcal{K}_Y(w, z)\|_{Y^+} \leq \bar{C}_{YX} h \|x - w\|_{X^+} + \bar{L}_{YX} \frac{h^2}{2} \|y - z\|_{Y^+}.$$

Similarly,

$$\begin{aligned}
& \|\mathcal{K}_X(x, y)(t) - \mathcal{K}_X(w, z)(t)\|_{X^+} \\
& \leq \int_0^h \int_{-\tau}^0 |C_{XX}(s + \sigma, \theta)| |V_X^+(x - w)(\sigma + \theta)| \, d\theta \, d\sigma \\
& \quad + \int_0^h \|L_{XY}(s + \sigma)\|_{\mathbb{R}^{d_X \leftarrow Y}} \|V_Y^+(y - z)_\sigma\|_Y \, d\sigma \\
& \leq \bar{C}_{XX} \int_0^h \|V_X^+(x - w)_\sigma\|_X \, d\sigma \\
& \quad + \int_0^h \|L_{XY}(s + \sigma)\|_{\mathbb{R}^{d_X \leftarrow Y}} \sigma \|y - z\|_{Y^+} \, d\sigma \\
& \leq \bar{C}_{XX} h \|x - w\|_{X^+} + \bar{L}_{XY} \frac{h^2}{2} \|y - z\|_{Y^+},
\end{aligned}$$

where

$$\bar{L}_{XY} := \sup_{t \in \mathbb{R}} \|L_{XY}(t)\|_{\mathbb{R}^{d_X \leftarrow Y}}$$

and

$$\bar{C}_{XX} := \operatorname{ess\,sup}_{(t, \theta) \in \mathbb{R} \times [-\tau, 0]} |C_{XX}(t, \theta)|.$$

Hence there exists $\bar{K} \geq 0$ such that

$$\|\mathcal{K}(x, y) - \mathcal{K}(w, z)\|_{X^+ \times Y^+} \leq \bar{K} \|(x, y) - (w, z)\|_{X^+ \times Y^+}.$$

By choosing h small enough, $\mathcal{K}$ is a contraction of constant $\bar{K} < 1$, and, by the contraction mapping theorem ([Theorem 2.33](#)), $\mathcal{K}$ has a unique fixed point, which is a solution of (6.3) on $[0, h]$. The same reasoning can then be applied to $[h, 2h]$, $[2h, 3h]$ and so on, yielding a unique solution of (6.3) on $[0, +\infty)$. $\square$

Corollary 6.4. *If the hypotheses of [Theorem 6.1](#) hold, then the operator*

$$I_{X^+ \times Y^+} - \mathcal{F}_s V^+$$

is invertible with bounded inverse and [\(6.8\)](#) admits a unique solution in $X^+ \times Y^+$.

Proof. Given $(f, g) \in X^+ \times Y^+$, the equation

$$(I_{X^+ \times Y^+} - \mathcal{F}_s V^+)(w, z) = (f, g)$$

has a unique solution $(w, z) \in X^+ \times Y^+$ if and only if the initial value problem

$$\begin{cases} x(t) = \int_{-\tau}^0 C_{XX}(s+t, \theta) x(t+\theta) d\theta + L_{XY}(s+t) y_t + f(t), & \text{if } t \in (0, h], \\ y'(t) = \int_{-\tau}^0 C_{YX}(s+t, \theta) x(t+\theta) d\theta + L_{YY}(s+t) y_t + g(t), & \text{if } t \in [0, h], \\ x_0 = 0 \in X, \\ y_0 = 0 \in Y, \end{cases}$$

has a unique solution (x, y) in $X^\pm \times Y^\pm$, with (w, z) coinciding with (x, y') on $[0, h]$. This follows from [Theorem 6.1](#). So $I_{X^+ \times Y^+} - \mathcal{F}_s V^+$ is invertible and bounded and the bounded inverse theorem ([Theorem 2.19](#)) completes the proof. $\square$

6.3 DISCRETIZATION

Let M and N be positive integers, referred to as discretization indices. Define the partitions of the time intervals $[-\tau, 0]$ and $[0, h]$ as in [subsection 4.3.1](#).

As for the function spaces, we proceed as in [subsection 4.3.2](#) and [section 5.3](#) to define the discretized spaces $X_M \times Y_M$ and $X_N^+ \times Y_N^+$ and the operators $P_M, R_M, \mathcal{L}_M, P_N^+, R_N^+$ and $\mathcal{L}_N^+$ with the same caveat on choosing appropriate subspaces $\tilde{X} \subset X$ and $\tilde{X}^+ \subset X^+$ as in [section 5.3](#). The equalities [\(4.11\)](#) and [\(4.13\)](#) hold also in this setting. As in [subsection 4.3.2](#) when not ambiguous the restrictions to subspaces of the prolongation, restriction and Lagrange interpolation operators are denoted in the same way as the operators themselves. Also, to avoid a cumbersome notation we use the same symbols for the prolongation, restriction and Lagrange interpolation operators applied both to elements of the single function spaces and to elements of the product spaces.

Finally, discretize the operator T as $T_{M,N}: X_M \times Y_M \rightarrow X_M \times Y_M$ according to [subsection 4.3.3](#), obtaining

$$T_{M,N}(\Phi, \Psi) := R_M V(P_M(\Phi, \Psi), P_N^+(W^*, Z^*))_{h,}$$

where $(W^*, Z^*) \in X_N^+ \times Y_N^+$ is a solution of the fixed point equation

$$(W, Z) = R_N^+ \mathcal{F}_s V(P_M(\Phi, \Psi), P_N^+(W, Z)) \quad (6.9)$$

for the given $(\Phi, \Psi) \in X_M \times Y_M$, along with the further reformulation

$$T_{M,N} = T_M^{(1)} + T_{M,N}^{(2)} (I_{X_N^+} - U_N^{(2)})^{-1} U_{M,N}^{(1)}, \quad (6.10)$$

with definitions for the different finite-dimensional operators analogous to the ones in [subsection 4.3.3](#). We establish that (6.9) is well posed in [subsection 6.4.2](#). Again, (6.10) is the basis for the construction of the matrix representation of $T_{M,N}$ in [appendix A](#).

6.4 CONVERGENCE ANALYSIS

As noted at the beginning of this chapter, the proof of convergence in the coupled case is a combination of the proofs for RFDEs and REs. Therefore, most proofs are omitted.

6.4.1 Additional spaces and assumptions

Consider the Banach spaces X_C and X_C^+ with their norms $\|\cdot\|_{X_C^+}$ and $\|\cdot\|_{X_C}$ as defined in [subsection 5.4.1](#) (i.e., in a piecewise fashion if $h < \tau$) and recall [Remark 5.8](#). Consider the Banach spaces of AC functions Y_{AC} and Y_{AC}^+ with their norms $\|\cdot\|_{Y_{AC}}$ and $\|\cdot\|_{Y_{AC}^+}$ as defined in [subsection 4.4.1](#). Equip the product spaces $X_C \times Y$, $X \times Y_{AC}$ and $X_C^+ \times Y_{AC}^+$ with the norms $\|\cdot\|_{X_C \times Y}$, $\|\cdot\|_{X \times Y_{AC}}$ and $\|\cdot\|_{X_C^+ \times Y_{AC}^+}$ defined as in (6.1); with this choice they are Banach spaces.

The hypotheses on the discretization nodes in $[0, h]$ and on $\mathcal{F}_s$ and V and the involved function spaces are once again different with respect to RFDEs and REs. This is due to the interaction between the various parts of the proof and the different regularization properties of the RE and the RFDE parts of the equation. The list of hypotheses for coupled equations is the following:

(H6.1) the meshes $\{\Omega_N^+\}_{N>0}$ are the Chebyshev zeros

$$t_{N,n} := \frac{h}{2} \left(1 - \cos \left(\frac{(2n-1)\pi}{2N} \right) \right), \quad n \in \{1, \dots, N\},$$

(see [Remark 6.14](#));

(H6.2) the hypothesis of [Corollary 6.4](#) holds;

(H6.3) $\mathcal{F}_s V^+ : X^+ \times Y^+ \rightarrow X^+ \times Y^+$ has range contained in $X_C^+ \times Y_{AC}^+$ and $\mathcal{F}_s V^+ : X^+ \times Y^+ \rightarrow X_C^+ \times Y_{AC}^+$ is bounded.

(H6.4) $\mathcal{F}_{X,s} V^- : X \times Y \rightarrow X^+$ has range contained in X_C^+ and $\mathcal{F}_{X,s} V^- : X \times Y \rightarrow X_C^+$ is bounded.

(H6.5) $\mathcal{F}_{Y,s} V^- : X \times Y \rightarrow Y^+$ is such that $\mathcal{F}_{Y,s} V^-(X \times Y_{AC}) \subset Y_{AC}^+$ and $\mathcal{F}_{Y,s} V^-|_{X \times Y_{AC}} : X \times Y_{AC} \rightarrow Y_{AC}^+$ is bounded.

Similarly to (4.15) and (5.8), inspired by the relevant literature on applications of delay equations (see section 1.1 for some references), we choose the coupled RE/RFDE

$$\left\{ \begin{array}{l} x(t) = \sum_{k=1}^p \int_{-\tau_k}^{-\tau_{k-1}} C_{XX}^{(k)}(t, \theta) x(t + \theta) d\theta \\ \quad + A_{XY}(t) y(t) + \sum_{k=1}^p B_{XY}^{(k)}(t) y(t - \tau_k) \\ \quad + \sum_{k=1}^p \int_{-\tau_k}^{-\tau_{k-1}} C_{XY}^{(k)}(t, \theta) y(t + \theta) d\theta, \\ y'(t) = \sum_{k=1}^p \int_{-\tau_k}^{-\tau_{k-1}} C_{YX}^{(k)}(t, \theta) x(t + \theta) d\theta \\ \quad + A_{YY}(t) y(t) + \sum_{k=1}^p B_{YY}^{(k)}(t) y(t - \tau_k) \\ \quad + \sum_{k=1}^p \int_{-\tau_k}^{-\tau_{k-1}} C_{YY}^{(k)}(t, \theta) y(t + \theta) d\theta, \end{array} \right. \quad (6.11)$$

as a prototype equation, with $\tau_0 := 0 < \tau_1 < \dots < \tau_p := \tau$. Remark 5.5 applies also in this case to C_{XX} and C_{YX} . The following assumptions allow to fulfill hypotheses (H6.3), (H6.4) and (H6.5):

- A_{XY} and $B_{XY}^{(k)}$ are continuous;
- the assumptions of Proposition 2.7 hold for C_{XX} and $C_{XY}^{(k)}$;
- A_{YY} and $B_{YY}^{(k)}$ are absolutely continuous;
- the assumptions of Proposition 2.8 hold for C_{YX} and $C_{YY}^{(k)}$.

The boundedness of $\mathcal{F}_{Y,s}$ follows from hypothesis (H6.2) and that of $\mathcal{F}_{X,s}$ follows from the assumptions of Propositions 2.7 and 2.8.[†] The current implementation is based on (6.11) as well (see appendix A for more details).

6.4.2 Well-posedness of the collocation equation

With reference to (6.9), let $(\varphi, \psi) \in X \times Y$ and consider the collocation equation

$$(W, Z) = R_N^+ \mathcal{F}_s V((\varphi, \psi), P_N^+(W, Z)) \quad (6.12)$$

in $(W, Z) \in X_N^+ \times Y_N^+$. Observe that we need to assume that

$$\mathcal{F}_s V: (X \times Y) \times (X^+ \times Y^+) \rightarrow X^+ \times Y^+$$

has range in $\tilde{X}^+ \times Y^+$: hypotheses (H6.3) and (H6.4) imply that such a subspace $\tilde{X}^+ \subset X^+$ (containing X_C^+) exists.

The aim of this section is to show that (6.12) has a unique solution and to study its relation to the unique solution $(w^*, z^*) \in X^+ \times Y^+$ of (6.8). Using (6.6), the equations (6.8) and (6.12) can be rewritten, respectively, as

$$(I_{X^+ \times Y^+} - \mathcal{F}_s V^+)(w, z) = \mathcal{F}_s V^-(\varphi, \psi)$$

[†] Propositions 2.7 and 2.8 assume that the integrand function u is integrable: in this case, being u continuous, the assumptions can probably be relaxed.

and

$$(I_{X_N^+ \times Y_N^+} - R_N^+ \mathcal{F}_s V^+ P_N^+)(W, Z) = R_N^+ \mathcal{F}_s V^-(\varphi, \psi). \quad (6.13)$$

The following preliminary result concerns the operators

$$I_{\tilde{X}^+ \times Y^+} - \mathcal{L}_N^+ \mathcal{F}_s V^+ \upharpoonright_{\tilde{X}^+ \times Y^+} : \tilde{X}^+ \times Y^+ \rightarrow \tilde{X}^+ \times Y^+, \quad (6.14)$$

and

$$I_{X_N^+ \times Y_N^+} - R_N^+ \mathcal{F}_s V^+ P_N^+ : X_N^+ \times Y_N^+ \rightarrow X_N^+ \times Y_N^+. \quad (6.15)$$

Proposition 6.5. *If the operator (6.14) is invertible, then the operator (6.15) is invertible. Moreover, given $(\bar{W}, \bar{Z}) \in X_N^+ \times Y_N^+$, the unique solution $(\hat{w}, \hat{z}) \in \tilde{X}^+ \times Y^+$ of*

$$(I_{\tilde{X}^+ \times Y^+} - \mathcal{L}_N^+ \mathcal{F}_s V^+ \upharpoonright_{\tilde{X}^+ \times Y^+})(w, z) = P_N^+(\bar{W}, \bar{Z}) \quad (6.16)$$

and the unique solution $(\hat{W}, \hat{Z}) \in X_N^+ \times Y_N^+$ of

$$(I_{X_N^+ \times Y_N^+} - R_N^+ \mathcal{F}_s V^+ P_N^+)(W, Z) = (\bar{W}, \bar{Z}) \quad (6.17)$$

are related by $(\hat{W}, \hat{Z}) = R_N^+(\hat{w}, \hat{z})$ and $(\hat{w}, \hat{z}) = P_N^+(\hat{W}, \hat{Z})$.

Proof. Apply Proposition 2.18 with $U := \tilde{X}^+ \times Y^+$, $V := X_N^+ \times Y_N^+$, $A := \mathcal{F}_s V^+ \upharpoonright_{\tilde{X}^+ \times Y^+}$, $P := P_N^+$, $R := R_N^+$, recalling (4.13). $\square$

As observed above, the equation (6.12) is equivalent to (6.13), hence, by choosing

$$(\bar{W}, \bar{Z}) = R_N^+ \mathcal{F}_s V^-(\varphi, \psi), \quad (6.18)$$

it is equivalent to (6.17). Observe also that thanks to (4.13) the equation

$$(w, z) = \mathcal{L}_N^+ \mathcal{F}_s V((\varphi, \psi), (w, z)) \quad (6.19)$$

can be rewritten as

$$(I_{X^+ \times Y^+} - \mathcal{L}_N^+ \mathcal{F}_s V^+)(w, z) = \mathcal{L}_N^+ \mathcal{F}_s V^-(\varphi, \psi) = P_N^+ R_N^+ \mathcal{F}_s V^-(\varphi, \psi),$$

which is equivalent to (6.16) with the choice (6.18). Thus, by Proposition 6.5, if the operator (6.14) is invertible, then the equation (6.12) has a unique solution $(W^*, Z^*) \in X_N^+ \times Y_N^+$ such that

$$(W^*, Z^*) = R_N^+(w_N^*, z_N^*), \quad (w_N^*, z_N^*) = P_N^+(W^*, Z^*),$$

where $(w_N^*, z_N^*) \in X^+ \times Y^+$ is the unique solution of (6.19). Note for clarity that (6.18) implies $(w_N^*, z_N^*) = (\hat{w}, \hat{z})$ for $(\hat{w}, \hat{z})$ in Proposition 6.5.

So, now we show that under suitable assumptions the operator

$$I_{X^+ \times Y^+} - \mathcal{L}_N^+ \mathcal{F}_s V^+ : X^+ \times Y^+ \rightarrow X^+ \times Y^+ \quad (6.20)$$

is invertible, hence proving the invertibility of (6.14). Indeed, since (6.14) is surjective and it is reasonable to assume that

$$(I_{X^+ \times Y^+} - \mathcal{L}_N^+ \mathcal{F}_s V^+)(\tilde{X}^+ \times Y^+) \subset \tilde{X}^+ \times Y^+$$

(as we already observed in section 5.3 and thanks to hypotheses (H6.3) and (H6.4)), if (6.20) is invertible then also (6.14) is.

Proposition 6.6. *If hypotheses (H6.1), (H6.2) and (H6.3) hold, then there exists a positive integer N_0 such that, for any $N \geq N_0$, the operator (6.20) is invertible and*

$$\begin{aligned} & \| (I_{X^+ \times Y^+} - \mathcal{L}_N^+ \mathcal{F}_s V^+)^{-1} \|_{X^+ \times Y^+ \leftarrow X^+ \times Y^+} \\ & \leq 2 \| (I_{X^+ \times Y^+} - \mathcal{F}_s V^+)^{-1} \|_{X^+ \times Y^+ \leftarrow X^+ \times Y^+}. \end{aligned}$$

Moreover, for each $(\varphi, \psi) \in X \times Y$, (6.19) has a unique solution $(w_N^*, z_N^*) \in X^+ \times Y^+$ and

$$\begin{aligned} & \| (w_N^*, z_N^*) - (z^*, w^*) \|_{X^+ \times Y^+} \\ & \leq 2 \| (I_{X^+ \times Y^+} - \mathcal{F}_s V^+)^{-1} \|_{X^+ \times Y^+ \leftarrow X^+ \times Y^+} \| \mathcal{L}_N^+ (w^*, z^*) - (w^*, z^*) \|_{X^+ \times Y^+}, \end{aligned} \quad (6.21)$$

where $(w^*, z^*) \in X^+ \times Y^+$ is the unique solution of (6.8).

6.4.3 Convergence of the eigenvalues

As in the previous chapters, the operators $T_{M,N}$ and T cannot be compared directly, since they are defined on different spaces. Again, the line of the proof is to successively replace the finite-dimensional operator $T_{M,N}$ with finite-rank operators $\hat{T}_{M,N}$ and $\hat{T}_N$ defined on X (Propositions 6.7 and 6.8) and then compare $\hat{T}_N$ with T .

Restricting these operators was required for RFDEs and not for REs: in the coupled case, as one might expect, it is necessary to restrict them only in the second component, i.e., to $X \times Y_{AC}$. We will comment in more detail on the connection between the necessity of restricting and the regularization properties of $\mathcal{F}_s V$ in chapter 7.

The proof is concluded by applying once more results from spectral approximation theory [27] (Lemma 2.25), in Proposition 6.12 and Theorem 6.13, obtaining the desired convergence results.

The first step in the convergence proof is to introduce the finite-rank operator $\hat{T}_{M,N}$ on $X \times Y$ and show the relation between its spectrum and that of $T_{M,N}$. Notice that, as for REs, $\hat{T}_{M,N}$ must be defined on a Banach space contained in $\tilde{X} \times Y$, due to the use of R_M , hence the natural choice of $X_C \times Y$.

Proposition 6.7. *The finite-dimensional operator $T_{M,N}$ has the same nonzero eigenvalues, with the same geometric and partial multiplicities, of the operator*

$$\hat{T}_{M,N} := P_M T_{M,N} R_M|_{X_C \times Y}: X_C \times Y \rightarrow X_C \times Y.$$

Moreover, if $(\Phi, \Psi) \in X_M \times Y_M$ is an eigenvector of $T_{M,N}$ associated with a nonzero eigenvalue μ , then $P_M(\Phi, \Psi) \in X_C \times Y$ is an eigenvector of $\hat{T}_{M,N}$ associated with the same eigenvalue μ .

Proof. Apply Proposition 2.22 with $U := X_C \times Y$, $V := X_M \times Y_M$, $A := T_{M,N}$, $P := P_M$, $R := R_M$, recalling (4.11), since prolongations are polynomials, hence continuous. $\square$

Define the operator $\hat{T}_N: X \times Y \rightarrow X \times Y$ as

$$\hat{T}_N(\varphi, \psi) := V((\varphi, \psi), (w_N^*, z_N^*))_h,$$

where $(w_N^*, z_N^*) \in X^+ \times Y^+$ is the solution of the fixed point equation (6.19), which, under hypotheses (H6.1), (H6.2) and (H6.3), is unique thanks to Propositions 6.5 and 6.6. Observe that w_N^* and z_N^* are polynomials. Then, for $(\varphi, \psi) \in X_C \times Y$, by (4.24),

$$\begin{aligned}\hat{T}_{M,N}(\varphi, \psi) &= P_M T_{M,N} R_M(\varphi, \psi) \\ &= P_M R_M V(P_M R_M(\varphi, \psi), P_N^+(W^*, Z^*))_h \\ &= \mathcal{L}_M V(\mathcal{L}_M(\varphi, \psi), (w_N^*, z_N^*))_h \\ &= \mathcal{L}_M \hat{T}_N \mathcal{L}_M(\varphi, \psi),\end{aligned}$$

where $(W^*, Z^*) \in X_N^+ \times Y_N^+$ and $(w_N^*, z_N^*) \in X^+ \times Y^+$ are the solutions, respectively, of (6.9) applied to $(\Phi, \Psi) = R_M(\varphi, \psi)$ and of (6.19) with $\mathcal{L}_M(\varphi, \psi)$ replacing (φ, ψ) . These solutions are unique under hypotheses (H6.1), (H6.2) and (H6.3), thanks again to Propositions 6.5 and 6.6.

Now we show the relation between the spectra of $\hat{T}_{M,N}$ and $\hat{T}_N$.

Proposition 6.8. *Assume that hypotheses (H6.1), (H6.2) and (H6.3) hold and let $M \geq N \geq N_0$, with N_0 given by Proposition 6.6. Then the operator $\hat{T}_{M,N}$ has the same nonzero eigenvalues, with the same geometric and partial multiplicities and associated eigenvectors, of the operator $\hat{T}_N$.*

As anticipated above, similarly to the case of RFDEs, in order to achieve the desired convergence properties of Lagrange interpolation it is necessary to restrict the operators T and $\hat{T}_N$ in the second component. Thanks to Proposition 2.23, their spectral properties are preserved.

Proposition 6.9. *The operators T and $\hat{T}_N$ have the same nonzero eigenvalues, with the same geometric and partial multiplicities and associated eigenvectors, as their restrictions to $X \times Y_{AC}$.*

The key step to obtain the main convergence result is the proof of the norm convergence of $\hat{T}_N$ to T when both are restricted to $X \times Y_{AC}$. As for the other kinds of equations, we need the following lemma, which extends the results of Corollary 6.4 to $(I - \mathcal{F}_s V^+) \upharpoonright_{X_C^+ \times Y_{AC}^+}$.

Lemma 6.10. *If hypotheses (H6.2) and (H6.3) hold, then $(I - \mathcal{F}_s V^+) \upharpoonright_{X_C^+ \times Y_{AC}^+}$ is invertible with bounded inverse.*

Proposition 6.11. *If hypotheses (H6.1), (H6.2), (H6.3), (H6.4) and (H6.5) hold, then*

$$\left\| \hat{T}_N \upharpoonright_{X \times Y_{AC}} - T \upharpoonright_{X \times Y_{AC}} \right\|_{X \times Y_{AC} \leftarrow X \times Y_{AC}} \xrightarrow{N \rightarrow +\infty} 0.$$

Proof. Let $(\varphi, \psi) \in X \times Y_{AC}$ and let (w^*, z^*) and (w_N^*, z_N^*) be the solutions of the fixed point equations (6.8) and (6.19), respectively. Then

$$\begin{aligned}(\hat{T}_N - T)(\varphi, \psi) &= V((\varphi, \psi), (w_N^*, z_N^*))_h - V((\varphi, \psi), (w^*, z^*))_h \\ &= V^+((w_N^*, z_N^*) - (w^*, z^*))_h \\ &= (V_X^+(w_N^* - w^*)_h, V_Y^+(z_N^* - z^*)_h).\end{aligned}$$

Assuming hypotheses (H6.3), (H6.4) and (H6.5) and recalling that

$$(w^*, z^*) = \mathcal{F}_s V^+(w^*, z^*) + \mathcal{F}_s V^-(\varphi, \psi), \quad (6.22)$$

it is clear that $(w^*, z^*) \in X_C^+ \times Y_{AC}^+$. By the definitions of the various norms and by [Proposition 6.6](#), there exists a positive integer N_0 such that, for any $N \geq N_0$,

$$\begin{aligned}
& \|(\hat{T}_N - T)(\varphi, \psi)\|_{X \times Y_{AC}} \\
&= \|V_X^+(w_N^* - w^*)_h\|_X + \|V_Y^+(z_N^* - z^*)_h\|_{Y_{AC}} \\
&\leq \|w_N^* - w^*\|_{X^+} + \left\| \int_0^1 (z_N^* - z^*)(\sigma) d\sigma \right\|_{Y_{AC}^+} \\
&\leq \|w_N^* - w^*\|_{X^+} + h \left(1 + \frac{h}{2}\right) \|z_N^* - z^*\|_{Y^+} \\
&\leq \max \left\{ 1, h \left(1 + \frac{h}{2}\right) \right\} \|(w_N^*, z_N^*) - (w^*, z^*)\|_{X^+ \times Y^+} \\
&\leq 2 \max \left\{ 1, h \left(1 + \frac{h}{2}\right) \right\} \|(I_{X^+ \times Y^+} - \mathcal{F}_s V^+)^{-1}\|_{X^+ \times Y^+ \leftarrow X^+ \times Y^+} \\
&\quad \|(\mathcal{L}_N^+ - I) \upharpoonright_{X_C^+ \times Y_{AC}^+}\|_{X^+ \times Y^+ \leftarrow X_C^+ \times Y_{AC}^+} \|(w^*, z^*)\|_{X_C^+ \times Y_{AC}^+}.
\end{aligned} \tag{6.23}$$

Thus, recalling [\(4.25\)](#) and [\(5.26\)](#),

$$\|(\hat{T}_N - T)(\varphi, \psi)\|_{X \times Y_{AC}} \leq k(N) \|(w^*, z^*)\|_{X_C^+ \times Y_{AC}^+},$$

where $k(N)$ is a scalar function such that $k(N) \rightarrow 0$ as $N \rightarrow +\infty$. Being $(I - \mathcal{F}_s V^+) \upharpoonright_{X_C^+ \times Y_{AC}^+}$ invertible with bounded inverse, thanks to [Lemma 6.10](#), it follows that

$$(w^*, z^*) = ((I - \mathcal{F}_s V^+) \upharpoonright_{X_C^+ \times Y_{AC}^+})^{-1} \mathcal{F}_s V^-(\varphi, \psi).$$

Observe that

$$\|\mathcal{F}_{X,s} V^- \upharpoonright_{X \times Y_{AC}}\|_{X_C^+ \leftarrow X \times Y_{AC}} \leq \max\{1, C\} \|\mathcal{F}_{X,s} V^-\|_{X_C^+ \leftarrow X \times Y},$$

where $C > 0$ is given by [Theorem 2.6](#), and

$$\begin{aligned}
\|\mathcal{F}_s V^- \upharpoonright_{X \times Y_{AC}}\|_{X_C^+ \times Y_{AC}^+ \leftarrow X \times Y_{AC}} &= \|\mathcal{F}_{X,s} V^- \upharpoonright_{X \times Y_{AC}}\|_{X_C^+ \leftarrow X \times Y_{AC}} \\
&\quad + \|\mathcal{F}_{Y,s} V^- \upharpoonright_{X \times Y_{AC}}\|_{Y_{AC}^+ \leftarrow X \times Y_{AC}},
\end{aligned}$$

hence, thanks again to [hypotheses \(H6.4\)](#) and [\(H6.5\)](#),

$$\begin{aligned}
\|(w^*, z^*)\|_{X_C^+ \times Y_{AC}^+} &= \left\| ((I - \mathcal{F}_s V^+) \upharpoonright_{X_C^+ \times Y_{AC}^+})^{-1} \right\|_{X_C^+ \times Y_{AC}^+ \leftarrow X_C^+ \times Y_{AC}^+} \\
&\quad \|\mathcal{F}_s V^- \upharpoonright_{X \times Y_{AC}}\|_{X_C^+ \times Y_{AC}^+ \leftarrow X \times Y_{AC}} \|(\varphi, \psi)\|_{X \times Y_{AC}},
\end{aligned}$$

implying the thesis. $\square$

As in [chapters 4](#) and [5](#), the final convergence result is obtained thanks to results from spectral approximation theory, namely the ones summarized in [Lemma 2.25](#), and classic results in interpolation theory.

Proposition 6.12. *Assume that the [hypotheses \(H6.1\)](#), [\(H6.2\)](#), [\(H6.3\)](#), [\(H6.4\)](#) and [\(H6.5\)](#) hold. If $\mu \in \mathbb{C} \setminus \{0\}$ is an eigenvalue of $T \upharpoonright_{X \times Y_{AC}}$ with finite algebraic multiplicity v and ascent l , and Δ is a neighborhood of μ such that μ is the only eigenvalue of $T \upharpoonright_{X \times Y_{AC}}$ in Δ , then there exists a positive integer $N_1 \geq N_0$, with*

N_0 given by [Proposition 6.6](#), such that, for any $N \geq N_1$, $\hat{T}_N|_{X \times Y_{AC}}$ has in Δ exactly v eigenvalues $\mu_{N,j}$, $j \in \{1, \dots, v\}$, counting their multiplicities. Moreover, if for each $(\varphi, \psi) \in \mathcal{E}_\mu$, where $\mathcal{E}_\mu$ is the generalized eigenspace of $T|_{X \times Y_{AC}}$ associated with μ , the functions w^* and z^* that solve [\(6.8\)](#) are of class C^p , with $p \geq 1$, then

$$\max_{j \in \{1, \dots, v\}} |\mu_{N,j} - \mu| = o(N^{\frac{1-p}{T}}).$$

Theorem 6.13. Assume that [hypotheses \(H6.1\), \(H6.2\), \(H6.3\), \(H6.4\) and \(H6.5\)](#) hold. If $\mu \in \mathbb{C} \setminus \{0\}$ is an eigenvalue of T with finite algebraic multiplicity v and ascent l , and Δ is a neighborhood of μ such that μ is the only eigenvalue of T in Δ , then there exists a positive integer $N_1 \geq N_0$, with N_0 given by [Proposition 6.6](#), such that, for any $N \geq N_1$ and any $M \geq N$, $T_{M,N}$ has in Δ exactly v eigenvalues $\mu_{M,N,j}$, $j \in \{1, \dots, v\}$, counting their multiplicities. Moreover, if for each $(\varphi, \psi) \in \mathcal{E}_\mu$, where $\mathcal{E}_\mu$ is the generalized eigenspace of T associated with μ , the functions w^* and z^* that solve [\(6.8\)](#) are of class C^p , with $p \geq 1$, then

$$\max_{j \in \{1, \dots, v\}} |\mu_{M,N,j} - \mu| = o(N^{\frac{1-p}{T}}).$$

The comments of [Remark 4.15](#) on the further error contribution due to quadrature of the integrals in $\mathcal{F}_s$ apply also to this case. Comments similar to the ones of [Remark 5.17](#) on the attainability of the infinite order of convergence hold here as well.

Remark 6.14. Nodes other than those required by [hypothesis \(H6.1\)](#) may be used. Indeed, they are only asked to satisfy $\Lambda_N = o(N)$, the hypotheses of [Theorem 2.16](#) and the thesis of [Theorem 2.17](#). Anyway, here we assume [hypothesis \(H6.1\)](#) since these are the nodes we actually use in implementing the method. $\triangleleft$

7

REGULARIZATION EFFECTS IN THE CONVERGENCE PROOFS

In the previous chapters we pointed out several differences in the method and the convergence proof between RFDEs, REs and coupled equations. These differences also shed light on the importance of some elements of the proofs. In this brief chapter we collect some comments on these topics.

7.1 POINTWISE TERMS AND RESTRICTING THE OPERATORS

In [Remark 5.4](#) we discussed the generality of the chosen form [\(5.2\)](#) for the linear RE, especially in comparison with [\(4.15\)](#) and terms involving the value of the solution at given points. Aside from the issue of whether terms of that kind can be well defined, the presence of those terms in RFDEs and their absence in REs is responsible for one of the major differences between the two cases. We consider here only RFDEs and REs; similar consideration apply to the coupled case.

Indeed, the necessity of restricting the operators T and $\hat{T}_N$ in [Proposition 4.8](#) depends on the different regularization properties of $\mathcal{F}_s V^-$ attainable with the chosen prototype RFDE [\(4.15\)](#) and RE [\(5.8\)](#), reflected in [hypotheses \(H4.4\)](#) and [\(H5.4\)](#).

Let us compare the proofs of [Propositions 4.10](#) and [5.14](#). In both [\(4.28\)](#) and [\(5.28\)](#), in order to ensure the norm convergence of $\mathcal{L}_N^+$ to the identity, their domain must be restricted to a subspace of functions more regular than those in the codomain, hence the need for bounds on $\|z^*\|_{Y_{AC}^+}$ and $\|w^*\|_{X_C^+}$, respectively. As a consequence, those bounds, respectively [\(4.29\)](#) and [\(5.29\)](#), involve the norm of $\mathcal{F}_s V^-$ with codomain the respective subspace of more regular functions. In both cases V^- has almost the same definition; in particular it is the identity on $[-\tau, 0]$. Thus a possible regularization effect of $\mathcal{F}_s V^-$ must depend entirely on $\mathcal{F}_s$.

The prototype form of RFDEs [\(4.15\)](#), and thus $\mathcal{F}_s$ in typical cases, includes terms involving the value of the solution at the present time and at discrete delays, implying that the image of a function via $\mathcal{F}_s$ cannot be smoother than the function itself. Hence the need to restrict the domain of $\mathcal{F}_s V^-$ to Y_{AC} , obtaining a bound in terms of $\|\psi\|_{Y_{AC}}$, which in turn requires to restrict the operators T and $\hat{T}_N$.

For REs, on the other hand, the prototype equation [\(5.8\)](#) only has distributed delays, hence, with suitable hypotheses on the integration kernel, $\mathcal{F}_s$ can have a regularization effect and there is no need to restrict the domain of $\mathcal{F}_s V^-$ and in turn of T and $\hat{T}_N$.

As seen above, the operator $\mathcal{F}_s V^-$ is not required to have a regularization effect, in order to be able to conclude the proof, and indeed different prop-

erties are required in [hypotheses \(H4.4\)](#) and [\(H5.4\)](#). On the other hand the regularization effect of $\mathcal{F}_s V^+$ is essential to prove that the discretized problem is well posed, i.e., that the discrete fixed point equations [\(4.14\)](#) and [\(5.14\)](#) have a unique solution. Again, this depends on the necessity to obtain the norm convergence of $\mathcal{L}_N^+$ to the identity in the bounds [\(4.26\)](#) and [\(5.27\)](#) in the proofs of [Propositions 4.5](#) and [5.10](#).

Unlike V^- , the definition of V^+ depends on the type of equation, thus the regularization properties of $\mathcal{F}_s V^+$ do not necessarily depend on $\mathcal{F}_s$ only. Nevertheless, apart from the precise function spaces involved, [hypotheses \(H4.3\)](#) and [\(H5.3\)](#) require similar regularization effects.

For REs [hypothesis \(H5.3\)](#) is attainable for the same reasons as [hypothesis \(H5.4\)](#) (recall that V^+ is the identity on $[0, h]$, hence it does not regularize).

For RFDEs the presence of pointwise terms in $\mathcal{F}_s$ does not prevent the regularization effect required in [hypothesis \(H4.3\)](#) since V^+ is null on $[-\tau, 0]$ and an integration operator on $[0, h]$.

We already observed in [chapter 5](#) that [Proposition 5.14](#) is more general than [Proposition 4.10](#), not requiring the restriction of the operators T and $\hat{T}_N$. In light of the previous remarks, we can conclude that if the prototype RFDE [\(4.15\)](#) does have neither discrete delays nor the present time term, then there is no need to restrict the operators since $\mathcal{F}_s$ exhibits a regularization effect under suitable hypotheses on the integration kernels, and $\hat{T}_N$ converges to T in norm in the whole state space.

7.2 THE “REGULARIZATION HIERARCHY”

We observed in [subsection 6.4.3](#) that for coupled equations it is necessary to restrict the operators T and $\hat{T}_N$ only in their second component, i.e., to $X \times Y_{AC}$. This is consistent with the remarks in the previous section.

However, as anticipated in [chapter 6](#), in earlier versions of the proof of the convergence of $\hat{T}_N$ to T in norm, the operators were restricted to the subspace of absolutely continuous functions for X and a subspace of (piecewise) continuously differentiable functions for Y . As an effect, some steps of the proofs were more involved, especially some of the bounds on relevant norms, which in [chapter 6](#) are a combination of the RFDE and RE cases.

Indeed, the inequality [\(6.21\)](#) in [Proposition 6.6](#) is a bound only on the norms in X^+ and Y^+ , so it may not be useful when using other norms. As an example, if the operators are restricted to X_{AC} in their first component, then in [\(6.23\)](#) we need a bound on $\|w_N^* - w^*\|_{X_{AC}^+}$. Manipulating the relevant equations, such a bound may be

$$\begin{aligned} \|w_N^* - w^*\|_{X_{AC}^+} &\leq 2\|((I - \mathcal{F}_{X,s} V_X^+) \upharpoonright_{X_{AC}^+})^{-1}\|_{X_{AC}^+ \leftarrow X_{AC}^+} \\ &\quad \left(\|(\mathcal{L}_N^+ - I) \mathcal{F}_{X,s} V_Y^+(z_N^* - z^*)\|_{X_{AC}^+} \right. \\ &\quad \left. + \|\mathcal{F}_{X,s} V_Y^+(z_N^* - z^*)\|_{X_{AC}^+} + \|(\mathcal{L}_N^+ - I)w^*\|_{X_{AC}^+} \right). \end{aligned}$$

In order to attain the convergence of $\mathcal{L}_N^+$ to the identity, even more regularity is required for w^* , in this case (piecewise) continuously differentiable (see

section 7.3 below for relevant results on the convergence of Lagrange interpolation). In turn this requires the same regularity for the second component Y of the state, due to the presence of pointwise terms in $\mathcal{F}_{X,s}$ for the chosen prototype coupled equation (6.11), since in Proposition 6.11 we determine the regularity of w^* thanks to (6.22), i.e.,

$$(w^*, z^*) = \mathcal{F}_s V^+(w^*, z^*) + \mathcal{F}_s V^-(\varphi, \psi).$$

The example shows that the choice of the subspaces for restricting the operators T and $\hat{T}_N$ is not independent between the two components of the state. From this dependence emerges what we could call a “regularization hierarchy”, such that for each choice of subspace of X for the restriction, the choice for Y is constrained to subspaces of functions more regular than a certain regularity, which in turn is strictly more regular than the chosen subspace of X . Acceptable choices for the subspaces include $X \times Y_{AC}$ (as in chapter 6), $X_C \times Y_{AC}$ and $X_{AC} \times Y_{C_*^1}$, where by C_*^1 we mean functions that are continuously differentiable except for admitting jump discontinuities in the derivative at a fixed finite set of points in the domain.*†

The proof with the restriction to $X \times Y_{AC}$ presented in chapter 6 is surely sufficient for our purposes. However, the alternative restrictions and the remarks in this chapter help us better understand the interplay between the different pieces of the proof, as well as between the different roles of current time, discrete delays and distributed delays terms.

7.3 CONVERGENCE OF THE DERIVATIVE OF THE INTERPOLANT

This section collects some nonstandard results on the convergence of the derivative of the Lagrange interpolating polynomial to the derivative of the original function. As anticipated above, they may be needed in case of different choices in the restriction of the operators to function subspaces.

The first results show that it is possible to approximate an AC function f by a sequence of smooth functions whose derivatives approximate the derivative of f in L^1 .

Theorem 7.1 ([18, Corollary 4.23]). *Let $I = (a, b)$. The space of smooth functions with compact support $C_c^\infty(I, \mathbb{R}^d)$ is dense in $L^p(I, \mathbb{R}^d)$ for all $1 \leq p < \infty$.*

Corollary 7.2. *Let $I = (a, b)$ and $f \in AC(I, \mathbb{R}^d)$. There exists a sequence $\{f_n\}_{n \in \mathbb{N}} \subset C^\infty(I, \mathbb{R}^d)$ such that $\{f'_n\}_{n \in \mathbb{N}} \subset C_c^\infty(I, \mathbb{R}^d)$, $\|f_n - f\|_\infty \xrightarrow{n \rightarrow \infty} 0$ and $\|f'_n - f'\|_1 \xrightarrow{n \rightarrow \infty} 0$.*

* In fact, a finite set of discontinuities in the derivative must be allowed in some spaces used in the proofs. As shown in Proposition 2.8, the derivatives of the integral terms in $\mathcal{F}_s(u, v)$ include terms containing values of u and v at discrete points: if $u \in X^\pm$ is discontinuous, then the derivative of the solution is discontinuous at points corresponding to the discontinuities of u , similarly to how classic breaking points are propagated in the solution of an RFDE.

† As a side note, using Banach spaces of C^1 or C_*^1 functions requires estimates on the derivative of the interpolation error in order to prove the convergence, since the norm of those spaces is the sum of the uniform norm of the function and of its derivative. Useful results on such estimates can be found in [93].

Proof. From [Lemma 2.3](#) follows that

$$f(t) = f(a) + \int_a^t f'(\sigma) d\sigma$$

with $f' \in L^1(I, \mathbb{R}^d)$. Then, by [Theorem 7.1](#) there exists a sequence $\{g_n\}_{n \in \mathbb{N}} \subset C_c^\infty(I, \mathbb{R}^d)$ such that $\|g_n - f'\|_1 \xrightarrow{n \rightarrow \infty} 0$. It is now sufficient to define

$$f_n(t) := f(a) + \int_a^t g_n(\sigma) d\sigma. \quad \square$$

The next result is a corollary of a very general result on the mean convergence of derivatives of Lagrange interpolating polynomials on nodes that are roots of orthogonal polynomials associated with generalized Jacobi weight functions [[93](#), Theorem 1 and Corollary]. Here we specialize it for interpolation on Chebyshev zeros. Recall the definitions of [subsection 2.3.2](#).

Corollary 7.3. *Let $I = [-1, 1]$ and let $\mathcal{L}_N$ be the Lagrange interpolation operator relevant to Chebyshev zeros in I . If $f \in C^1(I, \mathbb{R}^d)$, then*

$$\|f' - (\mathcal{L}_N f)'\|_1 \leq c E_{N-2}(f') \xrightarrow{N \rightarrow \infty} 0,$$

where c is some positive constant.

Proof. The inequality follows from [[93](#), Corollary] by taking $w(x) = (1 - x^2)^{-\frac{1}{2}}$, $p = 1$ and $r = 1$. The limit follows from the continuity of f' and [Theorem 2.11](#). $\square$

Finally, [Proposition 7.4](#) extends the result of [Corollary 7.3](#) to AC functions with exactly one discontinuity in the derivative (extending to a finite number of discontinuities is straightforward).

Proposition 7.4. *Let $I = [-1, 1]$ and let $\mathcal{L}_N$ be the Lagrange interpolation operator relevant to Chebyshev zeros in I . Let $x \in I$ and let $f \in \text{AC}(I, \mathbb{R}^d)$ with $f' \in C(I \setminus \{x\}, \mathbb{R}^d)$. Then*

$$\|f' - (\mathcal{L}_N f)'\|_1 \xrightarrow{N \rightarrow \infty} 0.$$

Proof. Let $\{t_{N,n}\}_{n \in \{1, \dots, N\}}$ be the N -degree Chebyshev zeros. For each $N \in \mathbb{N} \setminus \{0\}$, the N -degree and the $(N+1)$ -degree nodes form disjoint sets, so there exists a strictly increasing sequence of positive integers $\{M_N\}_{N \in \mathbb{N} \setminus \{0\}}$ such that, for each $N \in \mathbb{N} \setminus \{0\}$ and each $n \in \{1, \dots, M_N\}$, $t_{M_N, n} \neq x$. Thus, without loss of generality, it can be assumed that, for each $N \in \mathbb{N} \setminus \{0\}$ and each $n \in \{1, \dots, N\}$, $t_{N, n} \neq x$.

Let N be a positive integer. There exist $a_N, b_N \in I$ with $a_N < b_N$ such that $x \in I_N := [a_N, b_N]$ and, for each $n \in \{1, \dots, N\}$, $t_{N, n} \notin I_N$. Let $g_N \in C^1(I_N, \mathbb{R}^d)$ such that $g_N(a_N) = f(a_N)$, $g_N(b_N) = f(b_N)$, $g'_N(a_N) = f'(a_N)$ and $g'_N(b_N) = f'(b_N)$ and let $f_N := f|_{I_N} - g_N$. Obviously $f_N \in \text{AC}(I_N, \mathbb{R}^d)$ and

$$f_N(a_N) = f_N(b_N) = f'_N(a_N) = f'_N(b_N) = 0.$$

By [Corollary 7.2](#) there exists a sequence $\{g_{N,k}\}_{k \in \mathbb{N}} \subset C^\infty(I_N, \mathbb{R}^d)$ such that $\{g'_{N,k}\}_{k \in \mathbb{N}} \subset C_c^\infty(I_N, \mathbb{R}^d)$ and

$$\|g_{N,k} - f_N\|_{C(I_N, \mathbb{R}^d)} \xrightarrow{k \rightarrow \infty} 0, \quad \|g'_{N,k} - f'_N\|_{L^1(I_N, \mathbb{R}^d)} \xrightarrow{k \rightarrow \infty} 0.$$

Define the sequence of functions $\{f_{N,k}\}_{k \in \mathbb{N}}$ as

$$f_{N,k}(t) = \begin{cases} f(t), & t \in I \setminus I_N, \\ g_{N,k}(t) + g_N(t), & t \in I_N. \end{cases}$$

It is easy to show that $\{f_{N,k}\}_{k \in \mathbb{N}} \subset C^1(I, \mathbb{R}^d)$ and

$$\|f'_{N,k} - f'\|_1 = \|g'_{N,k} - (f - g_N)'\|_{L^1(I_N, \mathbb{R}^d)} = \|g'_{N,k} - f'_N\|_{L^1(I_N, \mathbb{R}^d)} \xrightarrow[k \rightarrow \infty]{} 0.$$

Hence, for each positive integer N and each $\epsilon > 0$, there exists $N_1(N) \in \mathbb{N}$ such that, for each $k > N_1(N)$, $\|f'_{N,k} - f'\|_1 < \frac{\epsilon}{2}$.

For each positive integer N and each $n \in \{1, \dots, N\}$ and each $k \in \mathbb{N}$, $f_{N,k}(t_{N,n}) = f(t_{N,n})$, which implies $\mathcal{L}_N f_{N,k} = \mathcal{L}_N f$. Thus, by [Corollary 7.3](#), for each $\epsilon > 0$, there exists a positive integer N_2 such that, for each $N > N_2$ and each $k \in \mathbb{N}$, $\|f'_{N,k} - (\mathcal{L}_N f)'\|_1 < \frac{\epsilon}{2}$.

The thesis follows easily. □

The motivating goal of this thesis was to provide numerical tools for the stability analysis of periodic solutions of REs and coupled REs/RFDEs. The pseudospectral collocation method developed here is more general, as it can be applied to any evolution operator of generic linear REs and equations.

This allows to study also the stability of equilibria by computing the eigenvalues of the evolution operator $T(h, 0)$ of the autonomous linearized system for any $h > 0$, exploiting the relation

$$\mu = e^{\lambda h} \quad (8.1)$$

existing between characteristic roots λ (i.e., the eigenvalues of the infinitesimal generator) and characteristic multipliers μ (i.e., the eigenvalues of the evolution operator $T(h, 0)$). Thanks to the relation (8.1), a characteristic root has negative, null or positive real part if and only if the corresponding characteristic multipliers are respectively inside, on or outside the unit circle in the complex plane. Hence, for any $h > 0$, an equilibrium is locally asymptotically stable if the characteristic multipliers of the evolution operator $T(h, 0)$ of the linearized system are inside the unit circle; it is unstable if at least one of them lies outside. When using evolution operators to study the stability of equilibria, it is common to choose $h = \tau$, as we do in this chapter.

This chapter contains some numerical tests for the methods presented in chapters 5 and 6. Thanks to the generality of the method, we approximate the multipliers for equilibria and for periodic solutions (both exact and numerically approximated) of REs and coupled REs/RFDEs. In both cases we study the convergence rates of the errors with respect to known quantities, namely, the absolute value of the dominant multipliers for bifurcations happening at known values of the parameters and, for periodic solutions, the trivial multiplier 1 (recall Proposition 3.8). Bifurcation diagrams and stability charts are also shown and computational times are briefly discussed, as well.

All tests are performed with GNU Octave 4.0.3, on a machine with an Intel Core i5-6200U CPU and 8GB of RAM, running Linux (Ubuntu 17.04, Linux 4.10.0-37-generic x86_64 kernel).

It is not easy to devise appropriate tests comparing the numerical results with known properties of the equations. Indeed, both for equilibria and periodic solutions it is difficult in general to analytically derive the stability properties, with the additional difficulty for periodic solutions that typically the solution itself is not known, rendering such analyses impossible.

For this reason, the first tests we perform in sections 8.1 and 8.2 reproduce results on the stability of equilibria of REs and coupled equations that are obtained in [12]. There, the characteristic roots are approximated by discretizing the infinitesimal generator via pseudospectral differentiation. In

the same sections we attempt to study the stability of periodic orbits of the same models, by linearizing the equation around a periodic solution approximated by applying the MatCont* ODE bifurcation package [32] to the system of nonlinear ODEs obtained via [10]. The trivial multiplier 1 is approximated with an error of the order of 10^{-6} for the example in section 8.1 and, unfortunately, only of the order of 10^{-2} for the example in section 8.2: a possible explanation is discussed in the relevant sections.

Next, in section 8.3 we study the stability of equilibria and periodic solutions for a REs for which relevant branches of periodic solutions are explicitly known. We study also the stability of branches of periodic solutions that are approximated numerically with an ad hoc extension to REs of the collocation method of [43, 70], obtaining better results with respect to the previous sections.

Finally, in section 8.4 we study a coupled equation carefully constructed in order to have an explicitly known periodic solution.

In this chapter we assume that the theory of chapter 3 holds for REs and coupled REs/RFDEs, hence we draw conclusions on the stability of periodic solutions of nonlinear equations from the Floquet multipliers of the corresponding linearized equations, as in section 8.3 and in particular in Figure 8.7.

We also use concepts from bifurcation theory, such as transcritical, Hopf and period doubling bifurcations, sometimes explaining them informally in the text. For more complete references on bifurcation theory, see [55, 68, 87].

Note that the results on the stability of periodic solutions of REs and coupled REs/RFDEs presented here are in absolute the first numerical (and indeed practically possible) available results. The only exception is [11], where the method of chapter 5 was used for the first time together with the method of [10]. See section 8.3 for more details.

8.1 SCALAR RE: CANNIBALISM

Consider the caricatural egg cannibalism model

$$A(t) = \beta \int_{a_{\text{repr}}}^{a_{\text{max}}} A(t-a) e^{-A(t-a)} da, \quad (8.2)$$

with $\beta > 0$ and $0 < a_{\text{repr}} < a_{\text{max}}$.

It has the equilibria $\bar{A}_0 = 0$ and

$$\bar{A}_1 = \log[\beta(a_{\text{max}} - a_{\text{repr}})],$$

which is biologically meaningful (i.e., nonnegative) if $\beta(a_{\text{max}} - a_{\text{repr}}) \geq 1$. For given a_{max} the curve $\beta = (a_{\text{max}} - a_{\text{repr}})^{-1}$ is the locus of transcritical bifurcations (where two equilibria exchange their stability properties). The curve

$$\begin{cases} a_{\text{repr}}(\omega) = \frac{2\pi}{\omega} - a_{\text{max}}, \\ \beta(\omega) = \frac{1}{2(a_{\text{max}} - \pi/\omega)} \exp\left(1 - \frac{\omega a_{\text{max}} - \pi}{\sin(\omega a_{\text{max}})}\right), \end{cases}$$

* <https://matcont.sourceforge.io/>

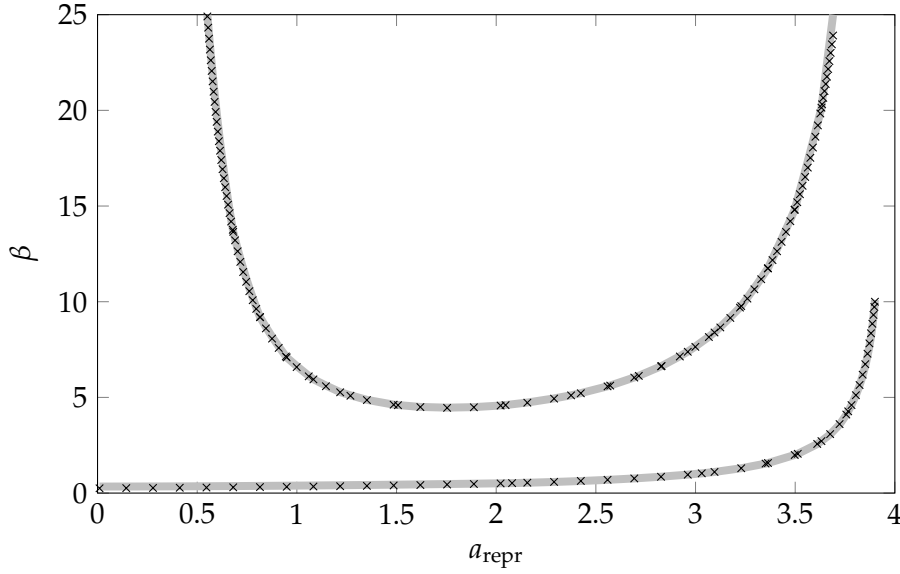


Figure 8.1: Stability chart for (8.2) with $a_{\max} = 4$ and $M = N = 10$. The thick gray lines are exact, the black crosses are the numerically obtained values.

parametrized by ω , is the locus of Hopf bifurcations (where a periodic solution arises from an equilibrium and exchanges the stability properties with it). Below the transcritical curve the trivial equilibrium $\bar{A}_0$ is asymptotically stable, while $\bar{A}_1$ is unstable. Above that curve $\bar{A}_0$ is unstable, and $\bar{A}_1$ is asymptotically stable between that curve and the Hopf curve, and unstable above the latter. On the transcritical curve the dominant multiplier of the equation linearized around the nontrivial equilibrium is 1, while on the Hopf curve the dominant multipliers are a complex conjugate pair on the unit circle. The linearized equation reads

$$A(t) = \frac{1 - \log[\beta(a_{\max} - a_{\text{repr}})]}{a_{\max} - a_{\text{repr}}} \int_{a_{\text{repr}}}^{a_{\max}} A(t - a) da. \quad (8.3)$$

Observe that (8.3) corresponds to (5.8) with

$$p = 2, \quad \tau_1 = -a_{\text{repr}}, \quad \tau_2 = -a_{\max}, \\ C^{(1)}(t, \theta) \equiv 0, \quad C^{(2)}(t, \theta) \equiv \frac{1 - \log[\beta(a_{\max} - a_{\text{repr}})]}{a_{\max} - a_{\text{repr}}}.$$

The model (8.2) has been studied in [12, section 5.1], where more details on the derivations can be found. There the analytic results were confirmed with the pseudospectral differentiation method for the infinitesimal generator presented in that paper.

Figure 8.1 shows the stability chart for (8.2), depicting both the exact curves and the ones obtained with the method of chapter 5 and standard zero-finding routines (e.g., MATLAB's `fzero`) to detect, respectively, the eigenvalue crossing the unit circle through 1 (transcritical bifurcation) and the complex conjugate pair crossing the unit circle (Hopf bifurcation). It is evident that the numerical method accurately reproduces the theoretical findings.

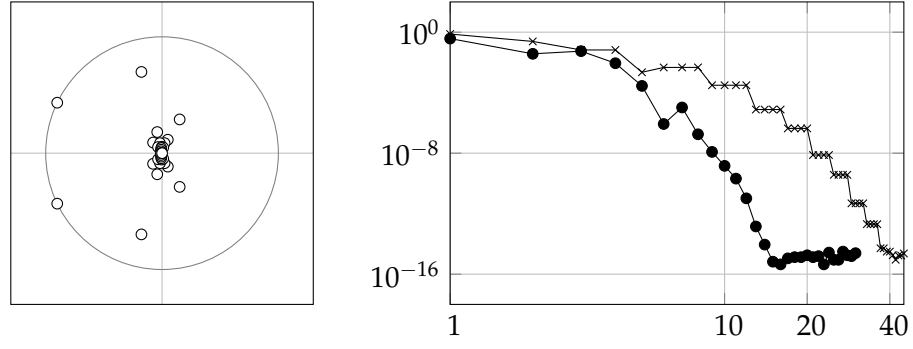


Figure 8.2: Numerical test for (8.2) with $a_{\max} = 4$, $a_{\text{repr}} = 3$ and $\beta = \frac{1}{2} \exp(1 + \frac{2\pi}{3\sqrt{3}})$. Left: eigenvalues of $T(a_{\max}, 0)$ for $M = N = 30$ with respect to the unit circle. Right: error with respect to 1 of the absolute value of the dominant eigenvalues of $T(a_{\max}, 0)$ (●) and error on the 0 real part of the rightmost characteristic roots obtained with the method of [12] (×), varying $M = N$.

In Figure 8.2 our method is compared with the method of [12] at a Hopf bifurcation point by computing the eigenvalues of $T(a_{\max}, 0)$. The comparison is made possible by the relation (8.1). For the sake of possible comparisons, the 12 dominant (largest in absolute value) multipliers obtained with $M = N = 30$ are

$$\begin{array}{ll}
 -0.900\,968\,867\,902\,422 & \pm \quad i \, 0.433\,883\,739\,117\,558 \\
 -0.177\,116\,911\,190\,809 & \pm \quad i \, 0.698\,197\,014\,147\,158 \\
 0.151\,441\,121\,920\,626 & \pm \quad i \, 0.289\,884\,717\,935\,784 \\
 -0.041\,332\,382\,067\,712 & \pm \quad i \, 0.181\,502\,421\,945\,093 \\
 0.049\,513\,635\,897\,963 & \pm \quad i \, 0.115\,568\,405\,490\,256 \\
 -0.077\,050\,693\,913\,191 & \pm \quad i \, 0.091\,458\,254\,492\,893
 \end{array}$$

The error plot shows the error with respect to 1 of the absolute value of the dominant multipliers for the former method and the error with respect to 0 of the real part of the rightmost characteristic roots for the latter. Both methods exhibit an infinite order of convergence, but the convergence of the method of [12] is slower, probably due to larger error constants (and also to the particular choice of M used therein).

Consider now the linearization of (8.2) around a generic solution $\bar{A}(t)$, which reads

$$A(t) = \beta \int_{a_{\text{repr}}}^{a_{\max}} (1 - \bar{A}(t-a)) e^{-\bar{A}(t-a)} A(t-a) da. \quad (8.4)$$

Periodic solutions arising from Hopf bifurcations exists for values of the parameters (a_{repr}, β) above the upper curve in Figure 8.1, but they are not known analytically. By discretizing (8.2) according to [10] and applying MatCont to the resulting system of ODEs, we can compute such a periodic solution. The method of chapter 5 can then be applied to (8.4), paired with the numerically approximated solution, to study the local stability of the latter. The solution given by MatCont is interpolated with piecewise cubic Hermite polynomials.

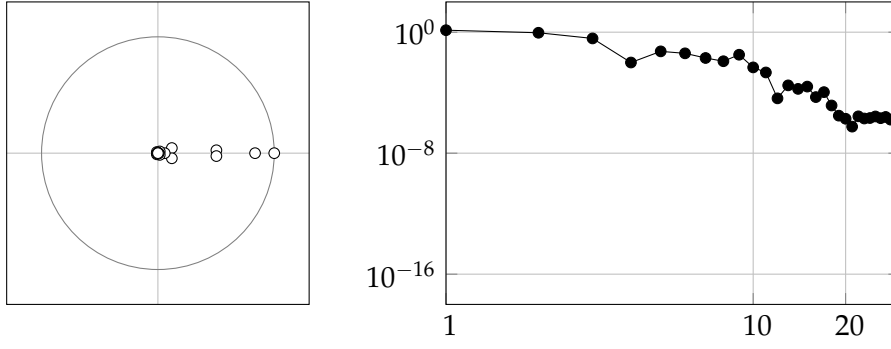


Figure 8.3: Numerical test for (8.2) with $a_{\max} = 4$, $a_{\text{repr}} = 3$ and $\beta = 7.99896953866859$ with a periodic solution of period $\Omega = 7.00000000009256$ computed with MatCont and [10]. Left: eigenvalues of $T(\Omega, 0)$ for $M = N = 30$ with respect to the unit circle. Right: error with respect to 1 of the dominant eigenvalue of $T(\Omega, 0)$, varying $M = N$.

Figure 8.3 shows the corresponding multipliers and the error on the trivial multiplier 1, varying $M = N$. Indeed, as seen in Proposition 3.8, the eigenvalue 1 is always present due to the linearization around a periodic solution. As anticipated, the results are qualitatively correct, but the errors stabilize at the order of 10^{-6} . This is probably due to the accuracy of the approximated periodic solution and to the interpolation, which is suggested also by better results obtained in section 8.3 with periodic solutions computed with a different method and a higher accuracy.

8.2 COUPLED RE/RFDE: SIMPLIFIED LOGISTIC DAPHNIA

The next example is a simplified version of the *Daphnia* model (1.3) with explicit terms for the survival probability, a fixed maturation age and a consumer-free resource dynamic of logistic type. The model is the coupled RE/RFDE

$$\begin{cases} b(t) = \beta S(t) \int_{a_{\text{repr}}}^{a_{\max}} b(t-a) da, \\ S'(t) = rS(t) \left(1 - \frac{S(t)}{K}\right) - \gamma S(t) \int_{a_{\text{repr}}}^{a_{\max}} b(t-a) da, \end{cases} \quad (8.5)$$

where all parameters are positive and $a_{\text{repr}} < a_{\max}$.

It has the equilibria $(\bar{b}_0, \bar{S}_0) = (0, 0)$, $(\bar{b}_1, \bar{S}_1) = (0, K)$, and

$$(\bar{b}_2, \bar{S}_2) = \left(\frac{r}{\gamma(a_{\max} - a_{\text{repr}})} \left(1 - \frac{1}{K\beta(a_{\max} - a_{\text{repr}})}\right), \frac{1}{\beta(a_{\max} - a_{\text{repr}})} \right).$$

The latter is biologically meaningful (i.e., nonnegative) if

$$\beta \geq \frac{1}{K(a_{\max} - a_{\text{repr}})}.$$

The trivial equilibrium $(\bar{b}_0, \bar{S}_0)$ is always unstable. The curve

$$\beta = \frac{1}{K(a_{\max} - a_{\text{repr}})}$$

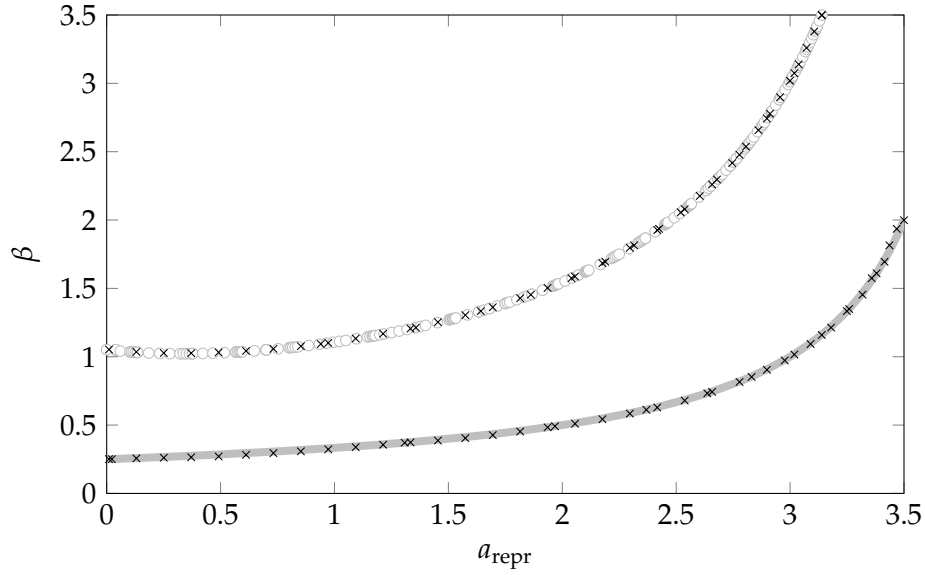


Figure 8.4: Stability chart for (8.5) with $a_{\max} = 4$, $\gamma = r = K = 1$ and $M = N = 10$. The thick gray line is the exact transcritical curve, the gray circles are the Hopf points obtained with the method of [12], the black crosses are the transcritical and Hopf points obtained with the method of chapter 6.

is the locus of transcritical bifurcations for the equilibria $(\bar{b}_1, \bar{S}_1)$ and $(\bar{b}_2, \bar{S}_2)$: below the curve, $(\bar{b}_1, \bar{S}_1)$ is asymptotically stable and $(\bar{b}_2, \bar{S}_2)$ is unstable; above the curve, $(\bar{b}_1, \bar{S}_1)$ is unstable, and $(\bar{b}_2, \bar{S}_2)$ is asymptotically stable between that curve and a certain curve to be determined numerically, where it undergoes a Hopf bifurcation, above which it is unstable.

As already noted in the previous example, on the transcritical curve the dominant multiplier of the equation linearized around the nontrivial equilibrium is 1, while on the Hopf curve the dominant multipliers are a complex conjugate pair on the unit circle. For an equilibrium $(\bar{b}, \bar{S})$ the linearized equation reads

$$\begin{cases} b(t) = \beta \bar{S} \int_{a_{\text{repr}}}^{a_{\max}} b(t-a) da + (a_{\max} - a_{\text{repr}}) \beta \bar{b} S(t), \\ S'(t) = -\gamma \bar{S} \int_{a_{\text{repr}}}^{a_{\max}} b(t-a) da + r \left(1 - 2 \frac{\bar{S}}{K}\right) S(t) \\ \quad - (a_{\max} - a_{\text{repr}}) \gamma \bar{b} S(t). \end{cases} \quad (8.6)$$

Observe that (8.6) corresponds to (6.11) with

$$\begin{aligned} p &= 2, & \tau_1 &= -a_{\text{repr}}, & \tau_2 &= -a_{\max}, \\ C_{XX}^{(2)}(t, \theta) &\equiv \beta \bar{S}, & A_{XY}(t) &\equiv (a_{\max} - a_{\text{repr}}) \beta \bar{b}, & C_{YX}^{(2)}(t, \theta) &\equiv -\gamma \bar{S}, \\ A_{YY}(t) &\equiv r \left(1 - 2 \frac{\bar{S}}{K}\right) - (a_{\max} - a_{\text{repr}}) \gamma \bar{b}, \end{aligned}$$

and all the other coefficients identically null.

The model (8.5) has been studied in [12, section 5.2], where more details on the derivations can be found.

Figure 8.4 shows the stability chart for (8.5), depicting the exact transcritical curve, the Hopf curve obtained with the method of [12], and both curves

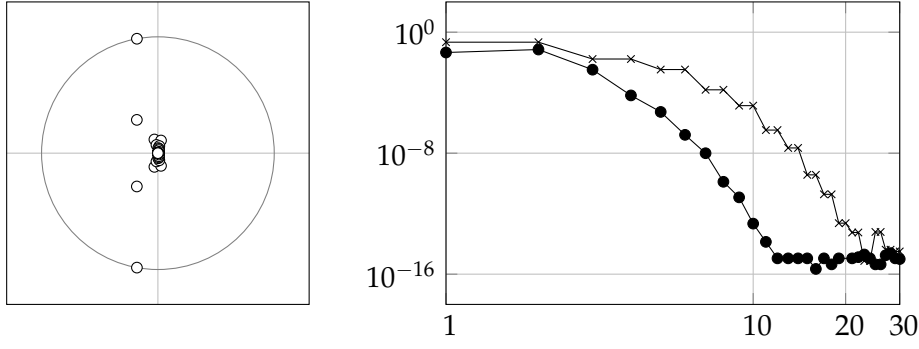


Figure 8.5: Numerical test for (8.5) with $a_{\max} = 4$, $\gamma = r = K = 1$, $a_{\text{repr}} = 2$ and the corresponding β determined with $M = N = 20$. Left: eigenvalues of $T(a_{\max}, 0)$ for $M = N = 20$ with respect to the unit circle. Right: error with respect to 1 of the absolute value of the dominant eigenvalues of $T(a_{\max}, 0)$ (●) and error on the 0 real part of the relevant rightmost characteristic roots obtained with the method of [12] (×), varying $M = N$.

obtained with the method of chapter 6 and standard zero-finding routines (e.g., MATLAB's `fzero`) to detect, respectively, the eigenvalue crossing the unit circle through 1 and the complex conjugate pair crossing the unit circle. It is evident again that the method of chapter 6 accurately reproduces both the theoretical and the numerical findings of [12].

As in the previous example, the error plot on the dominant multiplier shown in Figure 8.5 exhibits an infinite order of convergence, as predicted by Theorem 6.13 and typical of pseudospectral methods, and the convergence of the method of [12] is slower. The 12 dominant multipliers obtained with $M = N = 20$ are

$$\begin{array}{ll}
 -0.180\,485\,902\,767\,001 & \pm \quad i \, 0.983\,577\,571\,370\,139 \\
 -0.181\,056\,860\,044\,709 & \pm \quad i \, 0.286\,209\,800\,981\,762 \\
 -0.030\,117\,096\,937\,140 & \pm \quad i \, 0.117\,622\,033\,807\,973 \\
 0.026\,704\,875\,074\,216 & \pm \quad i \, 0.108\,886\,272\,641\,583 \\
 -0.012\,498\,302\,924\,958 & \pm \quad i \, 0.070\,932\,608\,242\,207 \\
 0.012\,218\,347\,106\,637 & \pm \quad i \, 0.061\,164\,935\,122\,025
 \end{array}$$

Again, a generic solution $(\bar{b}(t), \bar{S}(t))$ can be considered for the linearization, yielding

$$\begin{cases} b(t) = \beta \bar{S}(t) \int_{a_{\text{repr}}}^{a_{\max}} b(t-a) \, da + \beta S(t) \int_{a_{\text{repr}}}^{a_{\max}} \bar{b}(t-a) \, da, \\ S'(t) = -\gamma \bar{S}(t) \int_{a_{\text{repr}}}^{a_{\max}} b(t-a) \, da + r S(t) \left(1 - 2 \frac{\bar{S}(t)}{K}\right) \\ \quad - \gamma S(t) \int_{a_{\text{repr}}}^{a_{\max}} \bar{b}(t-a) \, da. \end{cases}$$

By discretizing (8.5) according to [10] and applying MatCont to the resulting system of ODEs, we can compute a periodic solution and study its stability with the method of chapter 6 applied to the resulting numerically linearized equation. Again, the solution given by MatCont is interpolated with piecewise cubic Hermite polynomials.

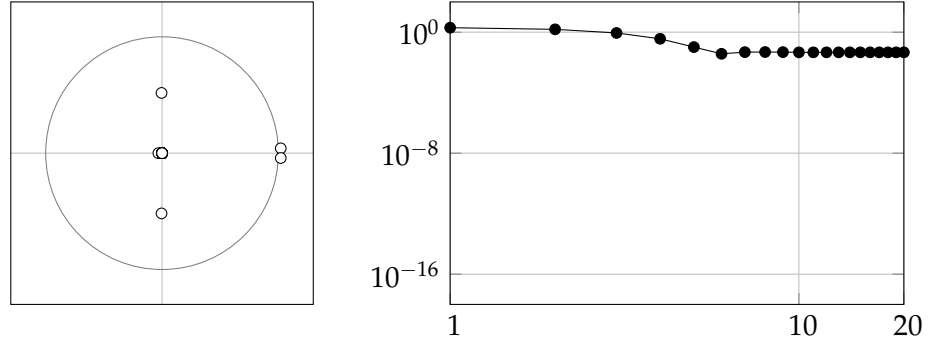


Figure 8.6: Numerical test for (8.5) with $a_{\max} = 4$, $\gamma = r = K = 1$, $a_{\text{repr}} = 2$ and $\beta = 3.02044380154012$ with a periodic solution of period $\Omega = 15.7721698691027$ computed with MatCont and [10]. Left: eigenvalues of $T(\Omega, 0)$ for $M = N = 20$ with respect to the unit circle. Right: error with respect to 1 of the dominant eigenvalue of $T(\Omega, 0)$, varying $M = N$.

Figure 8.6 shows the corresponding multipliers and the error on the trivial multiplier 1, varying $M = N$. Also in this case the results are qualitatively correct, but their accuracy is quite unsatisfactory with the errors stabilizing at the order of 10^{-2} , probably due to the accuracy of the interpolated numerical periodic solution obtained with MatCont.

8.3 SCALAR RE: SPECIAL RE WITH QUADRATIC NONLINEARITY

As a third example, we consider the scalar RE

$$x(t) = \frac{\gamma}{2} \int_1^3 x(t - \theta)(1 - x(t - \theta)) d\theta, \quad (8.7)$$

with $\gamma > 0$. It belongs to a class of nonlinear REs that has been studied in [11] both analytically and numerically. There it was established that the periodic solutions of period 4 arising from the Hopf bifurcation are characterized as solutions of a planar Hamiltonian system, hence providing a way to compute them with standard tools. However in the case of (8.7) those periodic solutions are explicitly known. Details on the derivations can be found in the cited paper.

Equation (8.7) has the equilibria $\bar{x}_0 = 0$ and $\bar{x}_1 = 1 - \frac{1}{\gamma}$, which is biologically meaningful for $\gamma \geq 1$. At $\gamma = \gamma_{\text{BP}} := 1$ the trivial equilibrium $\bar{x}_0$ exchanges its stability properties with the nontrivial equilibrium $\bar{x}_1$ in a transcritical bifurcation (also known as branching point), i.e., for $0 < \gamma < \gamma_{\text{BP}}$, $\bar{x}_0$ is asymptotically stable and $\bar{x}_1$ is unstable and negative, while for $\gamma > \gamma_{\text{BP}}$, $\bar{x}_0$ is unstable and $\bar{x}_1$ is positive, and it is stable for $\gamma < \gamma_{\text{H}} := 2 + \frac{\pi}{2}$. At $\gamma = \gamma_{\text{H}}$ a supercritical Hopf bifurcation occurs, i.e., for $\gamma > \gamma_{\text{H}}$, $\bar{x}_1$ is unstable and there exists a branch of periodic solutions which are asymptotically

stable for γ close enough to γ_H . These periodic solutions, all with period 4, are given by

$$\begin{cases} \bar{x}_2(t) = \frac{1}{2} + \frac{\pi}{4\gamma} + A \sin\left(\frac{\pi}{2}t\right), \\ A^2 = \frac{1}{2} - \frac{1}{\gamma} - \frac{\pi}{2\gamma^2}\left(1 + \frac{\pi}{4}\right), \end{cases} \quad (8.8)$$

as it can be straightforwardly verified. They have real values for $\gamma \geq \gamma_H$.

These theoretical findings on stability were confirmed in [11] by studying the characteristic roots of the system linearized around the equilibria with the method [12], and the characteristic multipliers of the system linearized around the periodic solutions with the method of chapter 5. By exploiting the relation (8.1), the results on characteristic roots can be reproduced using the method for characteristic multipliers. Given a solution $\bar{x}(t)$, the linearized system reads

$$x(t) = \frac{\gamma}{2} \int_1^3 (1 - 2\bar{x}(t - \theta))x(t - \theta) d\theta. \quad (8.9)$$

Observe that (8.9) corresponds to (5.8) with

$$\begin{aligned} p &= 2, & \tau_1 &= -1, & \tau_2 &= -3, \\ C^{(1)}(t, \theta) &\equiv 0, & C^{(2)}(t, \theta) &= \frac{\gamma}{2}(1 - 2\bar{x}(t + \theta)). \end{aligned}$$

The periodic solution $\bar{x}_2$ loses its stability at some $\gamma = \gamma_{PD1} > \gamma_H$, where a multiplier exiting the unit circle through -1 signals that a period doubling bifurcation occurs, i.e., a new branch of periodic solutions appears, with period roughly double the one of $\bar{x}_2$, asymptotically stable for values of γ close enough to the right of γ_{PD1} . The occurrence of a period doubling bifurcation was previously conjectured by O. Diekmann [11]: its numerical detection is an important positive result, confirming the validity of the conjecture.

Both the value γ_{PD1} and the new branch of solutions need to be computed numerically, the former with the method of chapter 5 and the latter with an adaptation to REs of the method of [43, 70]. Applying the same techniques to the new branch of solutions, we can find a second period doubling bifurcation at $\gamma = \gamma_{PD2} > \gamma_{PD1}$, and then, following the branch of periodic solutions arising from there, a third one at $\gamma = \gamma_{PD3} > \gamma_{PD2}$, suggesting the presence of a period doubling cascade.

The theoretical and numerical findings are summarized in the bifurcation diagram in Figure 8.7. The values of γ_{PD1} , γ_{PD2} and γ_{PD3} are determined using standard zero-finding routines (e.g., MATLAB's `fzero`) to detect an eigenvalue crossing the unit circle through -1 . An animated depiction of the stable solution (equilibrium or periodic solution) and the relevant characteristic roots and Floquet multipliers as γ varies along the bifurcation diagram can be seen in the movie `quadratic.mp4`[†] accompanying the paper [11].

Figure 8.8 shows the computed multipliers and the error on the trivial multiplier 1, which again is present due to the linearization. It concerns the exact periodic solution (8.8) at $\gamma_H < \gamma = 4.2 < \gamma_{PD1}$. The 12 dominant multipliers obtained with $M = N = 30$ are

[†] <http://www.math.u-szeged.hu/ejqtde/upload/supplements/id5273/quadratic.mp4>

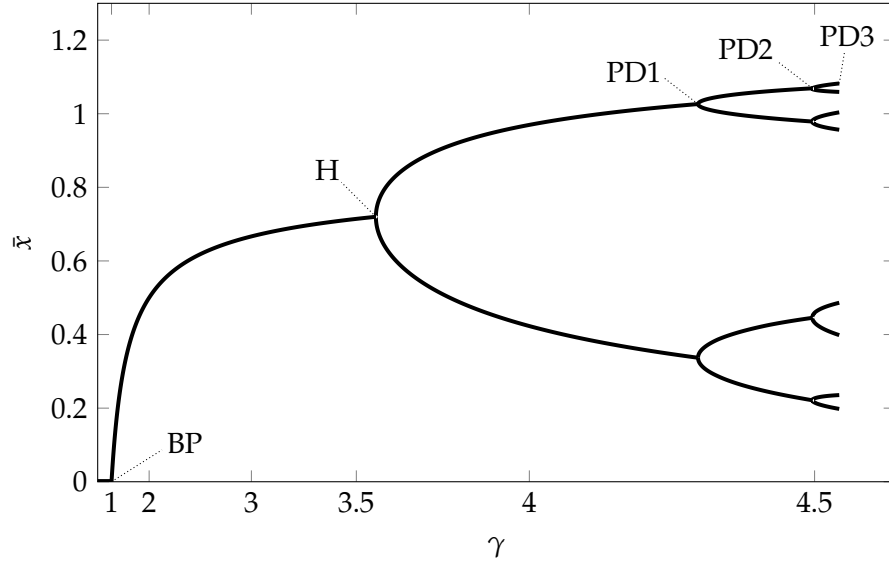


Figure 8.7: Bifurcation diagram for (8.7), depicting the value (for equilibria) and the extremal values (for cycles) of stable solutions.

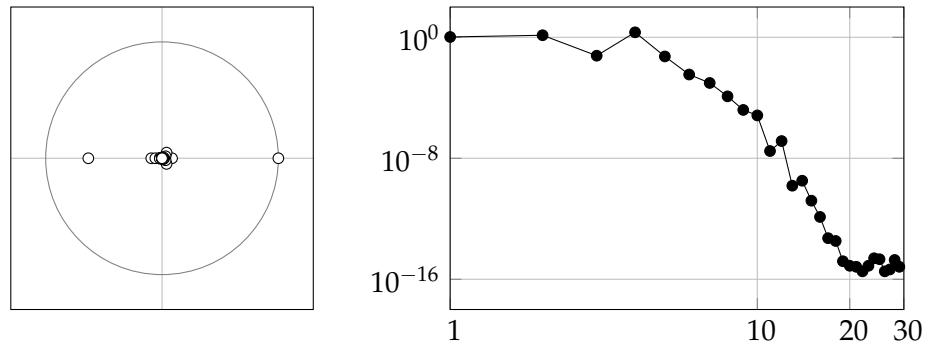


Figure 8.8: Numerical test for (8.7) with $\gamma_H < \gamma = 4.2\gamma_{PD1}$, linearized around (8.8). Left: eigenvalues of $T(4,0)$ for $M = N = 30$ with respect to the unit circle. Right: error on the trivial eigenvalue 1 of $T(4,0)$, varying $M = N$.

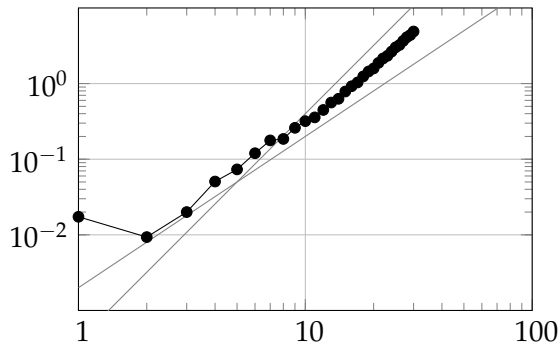


Figure 8.9: Computational times for Figure 8.8, varying $M = N$, with respect to a quadratic and a cubic monomials (gray lines). Values are the arithmetic means of three repetitions of the computations.

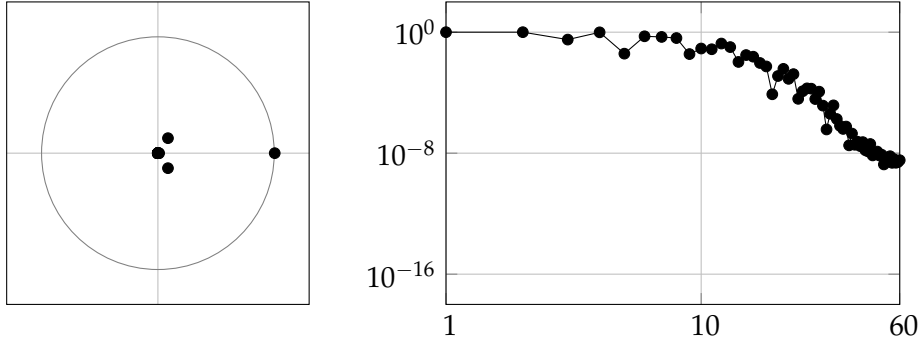


Figure 8.10: Numerical test for (8.7) with $\gamma_{PD1} < \gamma = 4.4 < \gamma_{PD2}$, linearized around a numerically approximated periodic solution of period $\Omega \approx 8.0189$. Left: eigenvalues of $T(\Omega, 0)$ for $M = N = 20$ with respect to the unit circle. Right: error on the trivial eigenvalue 1 of $T(\Omega, 0)$, varying $M = N$.

1	
−0.633 600 225 751 833	
−0.092 608 588 248 984	
0.086 089 716 020 891	
0.038 890 824 005 128	$\pm i$ 0.048 367 372 646 011
−0.057 497 449 216 221	
0.030 656 634 236 222	$\pm i$ 0.016 855 717 750 554
−0.018 895 793 736 796	$\pm i$ 0.003 622 911 724 905
0.018 828 814 030 643	

The error plot exhibits again an infinite order of convergence.

Figure 8.9 shows the computational times varying $M = N$ in comparison with quadratic and cubic monomials, which suggest that computational times depend on a power of $M = N$ with exponent between 2 and 3 (see section 8.5 below).

Figure 8.10 shows the computed multipliers and the error on the trivial multiplier 1 for $\gamma_{PD1} < \gamma = 4.4 < \gamma_{PD2}$ with a numerically approximated periodic solution. As opposed to sections 8.1 and 8.2, where the periodic solutions are computed by resorting to [10] and MatCont, here the periodic solution is obtained by extending the ideas of the collocation method of [43, 70] to REs (see chapter 9). With respect to the previous sections the achieved accuracy is much better, exhibiting one more time an infinite order of convergence.

Nevertheless, a number of nodes more than double is required to achieve the same accuracy as in the former case. Indeed, pseudospectral methods usually display a slower convergence for an increased length of the discretization interval (although still of infinite order). This can be justified by the properties of interpolation, since both the length of the interpolation interval (in this case the period of the solution) and bounds on the derivatives of the interpolated function (which are related to the number of oscillations) contribute to the error (recall Theorems 2.11 and 2.14): in this case, after the period doubling bifurcation both are roughly double than before. Moreover, the error includes contributions also from the approximation of the solution.

In [11] the pseudospectral discretization method presented in [10] is applied to the nonlinear equation as well, obtaining a system of nonlinear ODEs which was studied with MatCont, yielding comparable results. This was especially important at the time as a mutual validation of the methods of [10] and chapter 5, since the convergence proof for the latter was not complete and the one for the former concerned only equilibria. Indeed, [10] represents the first thorough bifurcation study of a RE.

8.4 COUPLED RE/RFDE: MODEL WITH EXACT PERIODIC SOLUTION

The final example of this chapter has been constructed ad hoc to have an analytically known periodic solution, so to avoid the errors in approximating it. It concerns the coupled equation

$$\begin{cases} x(t) = -\frac{1}{2} \left[\int_0^{\frac{7}{2}\pi} x(t-\sigma) d\sigma - \int_0^{\frac{\pi}{2}} \ln(y(t-\sigma)) d\sigma \right], \\ y'(t) = -\ln\left(y\left(t - \frac{\pi}{2}\right)\right)y(t). \end{cases} \quad (8.10)$$

The relevant periodic solution is

$$(\bar{x}(t), \bar{y}(t)) = (\sin(t), e^{\sin(t)}). \quad (8.11)$$

Observe that the period is 2π and that $\bar{x}(t) = \ln(\bar{y}(t))$. Of course (8.10) is not a realistic model, as it has been constructed specifically to have that simple exact periodic solution.

The linearization of (8.10) around the solution $(\bar{x}(t), \bar{y}(t))$ reads

$$\begin{cases} x(t) = -\frac{1}{2} \left[\int_0^{\frac{7}{2}\pi} x(t-\sigma) d\sigma - \int_0^{\frac{\pi}{2}} \frac{y(t-\sigma)}{\bar{y}(t-\sigma)} d\sigma \right], \\ y'(t) = -\ln\left(\bar{y}\left(t - \frac{\pi}{2}\right)\right)y(t) - \frac{y(t - \frac{\pi}{2})}{\bar{y}(t - \frac{\pi}{2})} \bar{y}(t). \end{cases} \quad (8.12)$$

It corresponds to (6.11) with

$$\begin{aligned} p &= 2, & \tau_1 &= -\frac{\pi}{2}, & \tau_2 &= -\frac{7}{2}\pi, \\ C_{XX}^{(1)}(t, \theta) &= C_{XX}^{(2)}(t, \theta) \equiv -\frac{1}{2}, & C_{XY}^{(1)}(t, \theta) &= \frac{1}{2\bar{y}(t+\theta)}, \\ A_{YY}(t) &= -\ln\left(\bar{y}\left(t + \frac{\pi}{2}\right)\right), & B^{(1)}(t) &= -\frac{\bar{y}(t)}{\bar{y}\left(t + \frac{\pi}{2}\right)}. \end{aligned}$$

Figure 8.11 shows the Floquet multipliers of (8.12) and the errors with respect to 1 of the two dominant multipliers. Indeed 1 is a multiplier, due to the linearization around a periodic solution. We find numerically a second eigenvalue 1. Both are approximated to the machine precision with an infinite order of convergence, hence we can expect that the ascent of the eigenvalue 1 is 1, i.e., that its algebraic and geometric multiplicities coincide and are equal to 2. The 12 dominant multipliers obtained with $M = N = 55$ are

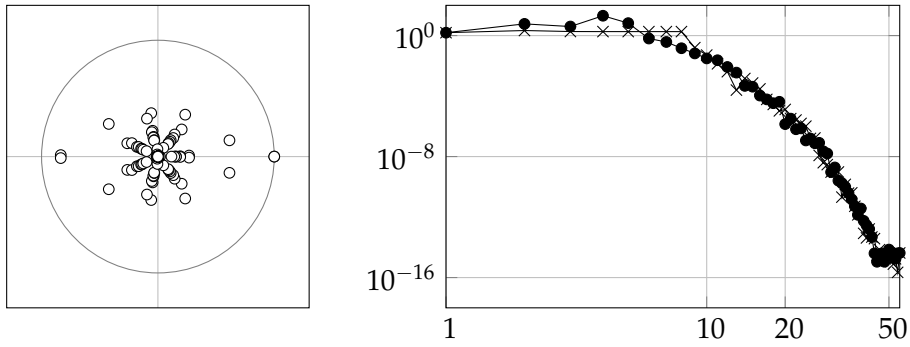


Figure 8.11: Numerical test for (8.10) linearized around (8.11). Left: eigenvalues of $T(2\pi, 0)$ for $M = N = 55$ with respect to the unit circle. Right: error on the two dominant eigenvalues of $T(2\pi, 0)$ (which happen to be both 1, with one of them the trivial one), varying $M = N$.

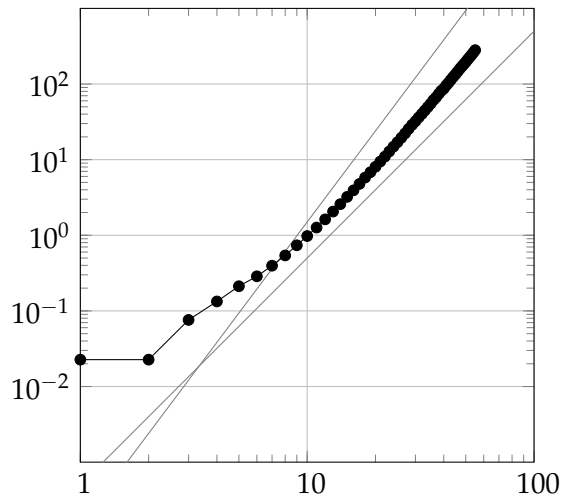


Figure 8.12: Computational times for Figure 8.11, varying $M = N$, with respect to monomials of degree 3 and 4 (gray lines). Values are the arithmetic means of three repetitions of the computations.

$$\begin{array}{rcl}
1.000\,000\,000\,000\,004 & \pm & i\,1.434\,187\,289\,235\,813 \times 10^{-15} \\
-0.835\,023\,280\,124\,518 & \pm & i\,0.012\,908\,862\,832\,415 \\
0.615\,089\,983\,054\,092 & \pm & i\,0.139\,694\,665\,978\,893 \\
-0.422\,313\,589\,924\,994 & \pm & i\,0.280\,698\,902\,212\,944 \\
0.234\,723\,865\,468\,381 & \pm & i\,0.360\,403\,056\,094\,859 \\
-0.057\,358\,337\,631\,699 & \pm & i\,0.373\,332\,749\,501\,845
\end{array}$$

Figure 8.12 shows the computational times varying $M = N$ in comparison with monomials of degrees 3 and 4, which suggest that computational times depend on a power of $M = N$ with exponent between 3 and 4.

8.5 COMPUTATIONAL TIMES

Figures 8.9 and 8.12 show the computational times varying $M = N$ for the application of the method to periodic solutions of the models of sections 8.3 and 8.4, respectively. In the first case the dependence seemed to be of order between 2 and 3, while in the second case between 3 and 4.

In fact this is lower than expected. The method involves the solution of a linear system for inverting $I_{X_N^+ \times Y_N^+} - U_N^{(2)}$, which has a cost of $O(N^3)$, and it requires the construction of $O(N)$ Lagrange coefficients and their evaluation $O(N^3)$ times, which cost, respectively, $O(N^2)$ and $O(N)$ each (see appendix A). Moreover, in the case of coupled REs/RFDEs the Lagrange coefficients are not only evaluated, but also integrated, which causes a further multiplication by $O(N)$. Hence, we would expect the computational time to be $O(N^4)$ for REs and $O(N^5)$ for coupled equations.

The lower order may be explained since the number of evaluations of Lagrange coefficients (or of their integrals) depends on the relative positions of the interpolation nodes and the delays. Moreover, in the current MATLAB/Octave implementation several computations are actually reused many times, thus lowering the computational cost.

In this final chapter, we discuss the problems that are left open in this thesis, along with some other research lines naturally arising in this context.

FLOQUET THEORY AND PRINCIPLE OF LINEARIZED STABILITY. The aim of this thesis is to provide numerical tools for studying the stability of periodic solutions of REs and coupled REs/RFDEs, based on linearization of the equation around a periodic solution and on the approximation of the eigenvalues of the monodromy operator of the linearized system by pseudospectral discretization.

This reasoning is theoretically based on Floquet theory and the principle of linearized stability, which are provided for RFDEs in the framework of sun-star calculus in [40] and summarized in [chapter 3](#). In [34] the sun-star calculus is extended to REs and coupled equations. In view of this, in [section 3.3](#) we discuss the extension of Floquet theory and the principle of linearized stability to REs. The arguments presented in that section are not complete and they tackle only [hypothesis \(H3.1\)](#), while for [hypothesis \(H3.2\)](#) we only present some considerations. Although also the numerical experiments hint in the direction of the validity of the theory for REs, a formal complete proof is still lacking and is the subject of ongoing research of the author and colleagues.

EIGENFUNCTIONS AND ORDER OF CONVERGENCE. The final convergence theorems [4.12](#), [5.16](#) and [6.13](#) guarantee that the eigenvalues of the discretized operator $T_{M,N}$ approximating an eigenvalue μ of the operator T converge to μ with an order depending on the smoothness of certain solutions. More precisely, in the case of coupled equations, if for each (φ, ψ) in the generalized eigenspace associated with μ the solution (w^*, z^*) of [\(6.8\)](#), i.e.

$$(w, z) = \mathcal{F}_s V((\varphi, \psi), (w, z)),$$

is of class C^p , then the error is $o(N^{\frac{1-p}{l}})$, with l the ascent of μ .

In the case of RFDEs, it is easy to show that solutions of the initial value problem with an eigenfunction of T as the initial value are of class C^∞ , then by induction on the rank the same is true for (linear combinations of) generalized eigenfunctions. Hence [Theorem 4.12](#) ensures an infinite order of convergence.

For REs, the relationship between the integration kernel $C(t, \theta)$ and the regularity of solutions is not clear, although we may expect solutions to be, in a certain way, as smooth as the kernel, similarly to [Propositions 2.7](#) and [2.8](#) and the results of [79] for convolution integrals. The characteristic shape of the error plots in [chapter 8](#) shows that in practice the infinite order of convergence can be achieved, suggesting that indeed the piecewise constant kernels may ensure that solutions are smooth. Investigating this issue for

REs and for coupled equations remains an open problem that the author wishes to study further in view of completing the relevant Floquet theory.

APPROXIMATION OF ORBITS. As mentioned in [chapter 1](#), a key objective in applications of delay equations is studying the stability of their solutions, which are seldom known explicitly, requiring to approximate them numerically. For periodic solutions in particular, this amounts to numerically solving boundary value problems (BVPs).

Different approaches to solving BVPs of delay equations are mentioned in [section 1.3](#), namely the recent abstract framework for BVPs of [\[76\]](#), the combination of standard methods for ODEs with the discretization technique of [\[10\]](#) and the collocation method for RFDEs of [\[43, 70\]](#). Appropriately extending the latter to REs and coupled equations is ongoing work of the author and colleagues.

REALISTIC MODELS. This thesis is part of a series of efforts to provide theoretical and numerical tools to study stability problems for delay equations, motivated by their importance in many fields, especially in mathematical biology. One of the goals of this research line is the stability and bifurcation analysis of the *Daphnia* model [\[38\]](#) briefly presented in [section 1.1](#). Applying the method of [chapter 6](#) to this important coupled RE/RFDE requires to tackle its many complications (most prominently state-dependent delays and external ODEs, see [section 1.1](#)) and is in the future plans of the author and colleagues.

It is worth noting that the only method currently available for studying stability and bifurcations of periodic solutions of *Daphnia*-like models is [\[10\]](#), although a proof of convergence for the periodic case is still ongoing research. The approach followed therein is in a certain sense opposite to the one adopted here. Instead of first linearizing the equation and then discretizing the linearized system, [\[10\]](#) discretizes directly the nonlinear system and then applies standard tools for bifurcation analysis and continuation opposed in (such as MatCont) to the resulting ODE system. This makes that approach simpler to use, since it allows to input the model in terms of the original equation without the need to linearize, and it makes use of widely used software packages for ODEs. Nevertheless, the approach of this thesis allows in general to attain a higher accuracy, being targeted to delay equations, at the expense of requiring a greater effort in writing the linearization and approximating the solution.

NEUTRAL DYNAMICS. In [chapter 5](#) we exclude REs involving evaluation of the solution at discrete points, since they are not well defined in terms of L^1 functions (recall [Remark 5.4](#)). However, REs with discrete delays appear in population models, as in the recent paper [\[35\]](#). Such equations give rise to neutral dynamics. Investigating them requires different theoretical tools with respect to [\[34, 40\]](#) and to the techniques used in the convergence proofs of [chapters 4, 5 and 6](#).

In recent years S.M. Verduyn Lunel and O. Diekmann have been working on a new perturbation theory for unbounded perturbations, corresponding to neutral delay equations, which would provide the needed theoretical

framework. As for the convergence proof, REs with discrete delays do not provide the required regularization effect, hence a different strategy would be needed.

LYAPUNOV EXPONENTS. As a last observation, recall that the method presented in [chapters 4, 5 and 6](#) can be applied not only to monodromy operators but to any evolution operator. Indeed, thanks to this generality, in [\[17\]](#) a discretization technique similar to that of [\[15\]](#) is applied to evolution operators of linear nonautonomous RFDEs in order to compute Lyapunov exponents. It is based on approximating the evolution operators by collocation and Fourier projection with finite-rank operators, then discretizing them into matrices and applying QR techniques (see, e.g., [\[33\]](#) and the references therein). This needs a notion of orthogonality in the state space, hence the method concerns initial value problems for RFDEs on a Hilbert state space of L^2 functions, instead of the usual Banach state space of continuous functions. It is interesting to note that for RFDEs the L^2 space contains the natural state space of continuous functions, while for REs it is contained in the natural L^1 state space.

In [\[11\]](#), the author and colleagues used the discretization proposed in [chapter 5](#) in the framework of [\[17\]](#), obtaining promising results in the computation of Lyapunov exponents. Investigating more precisely this extension remains in the interests of the author and colleagues.

This appendix is devoted to the implementation details concerning the method of [chapter 6](#) for coupled REs/RFDEs, including RFDEs and REs as special cases.

In [section A.1](#) we derive coefficients in the coupled case for the matrix representations of the finite-dimensional operators

$$\begin{aligned} T_M^{(1)}: X_M \times Y_M &\rightarrow X_M \times Y_M, & T_{M,N}^{(2)}: X_N^+ \times Y_N^+ &\rightarrow X_M \times Y_M, \\ U_{M,N}^{(1)}: X_M \times Y_M &\rightarrow X_N^+ \times Y_N^+, & U_N^{(2)}: X_N^+ \times Y_N^+ &\rightarrow X_N^+ \times Y_N^+, \end{aligned}$$

defined, respectively, as

$$\begin{aligned} T_M^{(1)}(\Phi, \Psi) &:= R_M(V^- P_M(\Phi, \Psi))_h, & T_{M,N}^{(2)}(W, Z) &:= R_M(V^+ P_N^+(W, Z))_h, \\ U_{M,N}^{(1)}(\Phi, \Psi) &:= R_N^+ \mathcal{F}_s V^- P_M(\Phi, \Psi), & U_N^{(2)}(W, Z) &:= R_N^+ \mathcal{F}_s V^+ P_N^+(W, Z), \end{aligned}$$

with the restriction and prolongation operators R_M , P_M , R_N^+ and R_N^+ defined as in [subsection 4.3.2](#). They allow to compute the coefficients of the matrix representation of the discretized evolution operator $T_{M,N}: X_M \times Y_M \rightarrow X_M \times Y_M$ as [\(6.10\)](#), i.e.,

$$T_{M,N} = T_M^{(1)} + T_{M,N}^{(2)}(I_{X_N^+ \times Y_N^+} - U_N^{(2)})^{-1} U_{M,N}^{(1)}.$$

Recalling [\(6.11\)](#), we consider as prototype model the coupled RE/RFDE

$$\left\{ \begin{aligned} x(t) &= \sum_{k=1}^p \int_{-\tau_k}^{-\tau_{k-1}} C_{XX}^{(k)}(t, \theta) x(t + \theta) d\theta \\ &\quad + A_{XY}(t) y(t) + \sum_{k=1}^p B_{XY}^{(k)}(t) y(t - \tau_k) \\ &\quad + \sum_{k=1}^p \int_{-\tau_k}^{-\tau_{k-1}} C_{XY}^{(k)}(t, \theta) y(t + \theta) d\theta, \\ y'(t) &= \sum_{k=1}^p \int_{-\tau_k}^{-\tau_{k-1}} C_{YX}^{(k)}(t, \theta) x(t + \theta) d\theta \\ &\quad + A_{YY}(t) y(t) + \sum_{k=1}^p B_{YY}^{(k)}(t) y(t - \tau_k) \\ &\quad + \sum_{k=1}^p \int_{-\tau_k}^{-\tau_{k-1}} C_{YY}^{(k)}(t, \theta) y(t + \theta) d\theta, \end{aligned} \right.$$

with $\tau_0 := 0 < \tau_1 < \dots < \tau_p := \tau$, which corresponds to defining the operator $\mathcal{F}_s: X^\pm \times Y^\pm \rightarrow X^+ \times Y^+$ as

$$\begin{aligned}\mathcal{F}_{X,s}(u,v)(t) &:= \sum_{k=1}^p \int_{-\tau_k}^{-\tau_{k-1}} C_{XX}^{(k)}(s+t, \theta) u(t+\theta) d\theta \\ &\quad + A_{XY}(s+t)v(t) + \sum_{k=1}^p B_{XY}^{(k)}(s+t)v(t-\tau_k) \\ &\quad + \sum_{k=1}^p \int_{-\tau_k}^{-\tau_{k-1}} C_{XY}^{(k)}(s+t, \theta) v(t+\theta) d\theta, \\ \mathcal{F}_{Y,s}(u,v)(t) &:= \sum_{k=1}^p \int_{-\tau_k}^{-\tau_{k-1}} C_{YX}^{(k)}(s+t, \theta) u(t+\theta) d\theta \\ &\quad + A_{YY}(s+t)v(t) + \sum_{k=1}^p B_{YY}^{(k)}(s+t)v(t-\tau_k) \\ &\quad + \sum_{k=1}^p \int_{-\tau_k}^{-\tau_{k-1}} C_{YY}^{(k)}(s+t, \theta) v(t+\theta) d\theta,\end{aligned}$$

for $t \in [0, h]$.

Then, in [section A.2](#) we briefly discuss the choices of interpolation nodes and of interpolation and quadrature formulas made in the current implementation,^{*} which is compatible both with MATLAB[†] [94] and GNU Octave[‡] [42]. In that implementation, after constructing the matrices, the eigenvalues of $T_{M,N}$ are computed with the standard function `eig` present in MATLAB and Octave.

A.1 MATRIX REPRESENTATION

We start by introducing some notations for block matrices. If $h \geq \tau$, for $(\Phi, \Psi) \in X_M \times Y_M$ and $m \in \{0, \dots, M\}$, denote as

$$[(\Phi, \Psi)]_m := (\Phi_{d_X m+1}, \dots, \Phi_{d_X(m+1)}, \Psi_{d_Y m+1}, \dots, \Psi_{d_Y(m+1)}),$$

the $(m+1)$ -th d_X -sized block of components of Φ and the $(m+1)$ -th d_Y -sized block of components of Ψ . If the pair (Φ, Ψ) is seen as the vector obtained by concatenating the components of Φ and Ψ , the components of $[(\Phi, \Psi)]_m$ are the ones of indices

$$(d_X m + 1, \dots, d_X(m+1), d_X(M+1) + d_Y m + 1, \dots, d_X(M+1) + d_Y(m+1)).$$

If $h < \tau$, instead, for $(\Phi, \Psi) \in X_M \times Y_M$, $q \in \{1, \dots, Q\}$ and $m \in \{0, \dots, M-1\}$ and for $q = Q$ and $m = M$, denote as

$$\begin{aligned}[(\Phi, \Psi)]_{q,m} &:= (\Phi_{d_X((q-1)M+m)+1}, \dots, \Phi_{d_X((q-1)M+m+1)}, \\ &\quad \Psi_{d_Y((q-1)M+m)+1}, \dots, \Psi_{d_Y((q-1)M+m+1)})\end{aligned}$$

^{*} To get the relevant MATLAB/Octave codes, visit the web pages listed on [page ii](#) or write to the author.

[†] <https://www.mathworks.com/products/matlab.html>

[‡] <https://www.gnu.org/software/octave/>

the $(m+1)$ -th d_X -sized block of components of the q -th block of Φ and the $(m+1)$ -th d_Y -sized block of components of the q -th block of Ψ . If the pair (Φ, Ψ) is seen as the vector obtained by concatenating the components of Φ and Ψ , the components of $[(\Phi, \Psi)]_{q,m}$ are the ones of indices

$$\begin{aligned} & (d_X((q-1)M+m)+1, \dots, d_X((q-1)M+m+1), \\ & d_X(QM+1)+d_Y((q-1)M+m)+1, \dots, \\ & d_X(QM+1)+d_Y((q-1)M+m+1)). \end{aligned}$$

Recall that $\Psi_M^{(q)} = \Psi_0^{(q+1)}$ for $q \in \{1, \dots, Q-1\}$, according to (4.10). Finally, for $(W, Z) \in X_N^+ \times Y_N^+$ and $n \in \{1, \dots, N\}$, denote as

$$[(W, Z)]_n := (W_{d_X(n-1)+1}, \dots, W_{d_X n}, Z_{d_Y(n-1)+1}, \dots, Z_{d_Y n})$$

the n -th d_X -sized block of components of W and the n -th d_Y -sized block of components of Z . If the pair (W, Z) is seen as the vector obtained by concatenating the components of W and Z , the components of $[(W, Z)]_n$ are the ones of indices

$$(d_X(n-1)+1, \dots, d_X n, d_X N + d_Y(n-1)+1, \dots, d_X N + d_Y n).$$

In the following, 0 denotes the scalar zero or a matrix of zeros of the dimensions implied by the context, while I_k denotes the identity matrix in $\mathbb{R}^{k \times k}$.

A.1.1 The matrix $T_M^{(1)}$

Let $(\Phi, \Psi) \in X_M \times Y_M$. If $h > \tau$, for $m \in \{0, \dots, M\}$

$$[T_M^{(1)}(\Phi, \Psi)]_m = (V^- P_M(\Phi, \Psi))_h(\theta_{M,m}) = V^- P_M(\Phi, \Psi)(h + \theta_{M,m}) = (0, \Psi_0),$$

since $h + \theta_{M,m} > 0$ and $P_M \Psi(0) = P_M \Psi(\theta_{M,0}) = \Psi_0$, hence

$$T_M^{(1)} = \begin{pmatrix} [T_M^{(1)}]_{XX} & [T_M^{(1)}]_{XY} \\ [T_M^{(1)}]_{YX} & [T_M^{(1)}]_{YY} \end{pmatrix} \in \mathbb{R}^{(d_X+d_Y)(M+1) \times (d_X+d_Y)(M+1)},$$

where

$$\begin{aligned} [T_M^{(1)}]_{XX} &= 0 \in \mathbb{R}^{d_X(M+1) \times d_X(M+1)}, \\ [T_M^{(1)}]_{XY} &= 0 \in \mathbb{R}^{d_X(M+1) \times d_Y(M+1)}, \\ [T_M^{(1)}]_{YX} &= 0 \in \mathbb{R}^{d_Y(M+1) \times d_X(M+1)}, \\ [T_M^{(1)}]_{YY} &= \begin{pmatrix} 1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \dots & 0 \end{pmatrix} \otimes I_{d_Y} \in \mathbb{R}^{d_Y(M+1) \times d_Y(M+1)}. \end{aligned}$$

If $h = \tau$, instead, for $m \in \{0, \dots, M-1\}$, $[T_M^{(1)}(\Phi, \Psi)]_m = (0, \Psi_0)$ as above. For $m = M$,

$$[T_M^{(1)}(\Phi, \Psi)]_M = V^- P_M(\Phi, \Psi)(h + \theta_{M,M}) = P_M(\Phi, \Psi)(\theta_{M,0}) = (\Phi_0, \Psi_0).$$

Thus $T_M^{(1)}$ is the same as above, except for

$$[T_M^{(1)}]_{XX} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \end{pmatrix} \otimes I_{d_X} \in \mathbb{R}^{d_X(M+1) \times d_X(M+1)}.$$

Finally, if $h < \tau$, for $m \in \{0, \dots, M-1\}$ and $q \in \{1, \dots, Q-1\}$,

$$\begin{aligned} [T_M^{(1)}(\Phi, \Psi)]_{q,m} &= V^- P_M(\Phi, \Psi)(h + \theta_{M,m}^{(q)}) \\ &= \begin{cases} (0, \Psi_0^{(1)}), & q = 1, \\ P_M(\Phi, \Psi)(\theta_{M,m}^{(q-1)}), & q \in \{2, \dots, Q-1\}, \end{cases} \\ &= \begin{cases} (0, \Psi_0^{(1)}), & q = 1, \\ (\Phi_m^{(q-1)}, \Psi_m^{(q-1)}), & q \in \{2, \dots, Q-1\}, \end{cases} \end{aligned}$$

while for $m \in \{0, \dots, M\}$ and $q = Q$,

$$\begin{aligned} [T_M^{(1)}(\Phi, \Psi)]_{Q,m} &= P_M(\Phi, \Psi)(h + \theta_{M,m}^{(Q)}) \\ &= \sum_{j=0}^M \ell_{M,j}^{(Q-1)}(h + \theta_{M,m}^{(Q)})(\Phi_j^{(Q-1)}, \Psi_j^{(Q-1)}). \end{aligned}$$

Observe that if $Qh = \tau$, then $[T_M^{(1)}(\Phi, \Psi)]_{Q,m} = (\Phi_m^{(Q-1)}, \Psi_m^{(Q-1)})$, since $h + \theta_{M,m}^{(Q)} = \theta_{M,m}^{(Q-1)}$. Then

$$T_M^{(1)} = \begin{pmatrix} [T_M^{(1)}]_{XX} & [T_M^{(1)}]_{XY} \\ [T_M^{(1)}]_{YX} & [T_M^{(1)}]_{YY} \end{pmatrix} \in \mathbb{R}^{(d_X+d_Y)(QM+1) \times (d_X+d_Y)(QM+1)},$$

where

$$\begin{aligned} [T_M^{(1)}]_{XY} &= \mathbf{0} \in \mathbb{R}^{d_X(QM+1) \times d_Y(QM+1)}, \\ [T_M^{(1)}]_{YX} &= \mathbf{0} \in \mathbb{R}^{d_Y(QM+1) \times d_X(QM+1)}, \end{aligned}$$

and

$$\begin{aligned} [T_M^{(1)}]_{XX} &\in \mathbb{R}^{d_X(QM+1) \times d_X(QM+1)}, \\ [T_M^{(1)}]_{YY} &\in \mathbb{R}^{d_Y(QM+1) \times d_Y(QM+1)} \end{aligned}$$

are given by

$$[T_M^{(1)}]_{XX} = \begin{pmatrix} 0 & & & & & & & & \\ \vdots & & & & & & & & \\ 0 & & & & & & & & \\ 1 & \cdots & 0 & & & & & & \\ \vdots & \ddots & \vdots & & & & & & \\ 0 & \cdots & 1 & & & & & & \\ & & & \ddots & & & & & \\ & & & & 1 & \cdots & 0 & & \\ & & & & \vdots & \ddots & \vdots & & \\ & & & & 0 & \cdots & 1 & & \\ & & & & & & & \Xi_{0,0} & \cdots & \Xi_{M-1,0} & \Xi_{M,0} & 0 & \cdots & 0 \\ & & & & & & & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ & & & & & & & & \vdots & & \Xi_{M,M-1} & 0 & \cdots & 0 \\ \Xi_{0,M-1} & \cdots & \Xi_{M-1,M-1} & \Xi_{M,M-1} & 0 & \cdots & 0 \\ & \Xi_{0,M} & \cdots & \Xi_{M-1,M} & \Xi_{M,M} & 0 & \cdots & 0 \end{pmatrix} \otimes I_{d_X},$$

$$[T_M^{(1)}]_{Y Y} = \begin{pmatrix} 1 & & & & & & & \\ \vdots & & & & & & & \\ 1 & & & & & & & \\ 1 & \cdots & 0 & & & & & \\ \vdots & \ddots & \vdots & & & & & \\ 0 & \cdots & 1 & & & & & \\ & & & \ddots & & & & \\ & & & & 1 & \cdots & 0 & \\ & & & & \vdots & \ddots & \vdots & \\ & & & & 0 & \cdots & 1 & \\ & & & & & & & \Xi_{0,0} & \cdots & \Xi_{M-1,0} & \Xi_{M,0} & 0 & \cdots & 0 \\ & & & & & & & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ & & & & & & & \Xi_{0,M-1} & \cdots & \Xi_{M-1,M-1} & \Xi_{M,M-1} & 0 & \cdots & 0 \\ & & & & & & & \Xi_{0,M} & \cdots & \Xi_{M-1,M} & \Xi_{M,M} & 0 & \cdots & 0 \end{pmatrix} \otimes I_{d_Y},$$

where $\Xi_{m,i} := \ell_{M,m}^{(Q-1)}(h + \theta_{M,i}^{(Q)})$ for $m, i \in \{0, \dots, M\}$ and missing entries are 0. In both matrices the order of rows and columns corresponds to the order of components in (4.12). Indeed they can be seen as block matrices with Q rows (respectively, columns), where the first $Q - 1$ consist of blocks of height (respectively, width) M and the last of blocks of height (respectively, width) $M + 1$. However, looking at the actual matrices, a slightly different block structure emerges: still $Q - 1$ rows of height M and a last row of height $M + 1$ can be seen, but there appear $Q - 2$ columns of width M followed by a column of width $M + 1$ and a last column of width M ; the top-left column (of zeros for $[T_M^{(1)}]_{XX}$ and of ones for $[T_M^{(1)}]_{YY}$) has height M , the identity blocks are I_M , the block of Lagrange coefficients has dimensions $(M + 1) \times (M + 1)$ and the bottom-right block of zeros has dimensions $(M + 1) \times M$. Note that if $Qh = \tau$ then $\ell_{M,j}^{(Q-1)}(h + \theta_{M,m}^{(Q)}) = \ell_{M,j}^{(Q-1)}(\theta_{M,m}^{(Q-1)}) = \delta_{m,j}$ and the block of Lagrange coefficients is actually I_{M+1} .

A.1.2 The matrix $T_{M,N}^{(2)}$

Let $(W, Z) \in X_N^+ \times Y_N^+$. If $h > \tau$, for $m \in \{0, \dots, M\}$,

$$\begin{aligned} [T_{M,N}^{(2)}(W, Z)]_m &= (V^+ P_N^+(W, Z))_h(\theta_{M,m}) \\ &= (V^+ P_N^+(W, Z))(h + \theta_{M,m}) \\ &= (P_N^+ W(h + \theta_{M,m}), \int_0^{h+\theta_{M,m}} P_N^+ Z(\sigma) \, d\sigma) \\ &= \sum_{n=1}^N \left(\ell_{N,n}^+(h + \theta_{M,m}) W_n, \int_0^{h+\theta_{M,m}} \ell_{N,n}^+(\sigma) \, d\sigma Z_n \right), \end{aligned}$$

hence

$$T_{M,N}^{(2)} = \begin{pmatrix} [T_{M,N}^{(2)}]_{XX} & [T_{M,N}^{(2)}]_{XY} \\ [T_{M,N}^{(2)}]_{YX} & [T_{M,N}^{(2)}]_{YY} \end{pmatrix} \in \mathbb{R}^{(d_X+d_Y)(M+1) \times (d_X+d_Y)N},$$

where

$$\begin{aligned} [T_{M,N}^{(2)}]_{XY} &= \mathbf{0} \in \mathbb{R}^{d_X(M+1) \times d_Y N}, \\ [T_{M,N}^{(2)}]_{YX} &= \mathbf{0} \in \mathbb{R}^{d_Y(M+1) \times d_X N}, \end{aligned}$$

and

$$\begin{aligned} [T_{M,N}^{(2)}]_{XX} &\in \mathbb{R}^{d_X(M+1) \times d_X N}, \\ [T_{M,N}^{(2)}]_{YY} &\in \mathbb{R}^{d_Y(M+1) \times d_Y N}, \end{aligned}$$

are given by

$$\begin{aligned} [T_{M,N}^{(2)}]_{XX} &= \begin{pmatrix} \ell_{N,1}^+(h + \theta_{M,0}) & \cdots & \ell_{N,N}^+(h + \theta_{M,0}) \\ \vdots & \ddots & \vdots \\ \ell_{N,1}^+(h + \theta_{M,M}) & \cdots & \ell_{N,N}^+(h + \theta_{M,M}) \end{pmatrix} \otimes I_{d_X}, \\ [T_{M,N}^{(2)}]_{YY} &= \begin{pmatrix} \int_0^{h+\theta_{M,0}} \ell_{N,1}^+(\sigma) d\sigma & \cdots & \int_0^{h+\theta_{M,0}} \ell_{N,N}^+(\sigma) d\sigma \\ \vdots & \ddots & \vdots \\ \int_0^{h+\theta_{M,M}} \ell_{N,1}^+(\sigma) d\sigma & \cdots & \int_0^{h+\theta_{M,M}} \ell_{N,N}^+(\sigma) d\sigma \end{pmatrix} \otimes I_{d_Y}. \end{aligned}$$

If $h = \tau$, instead, for $m \in \{0, \dots, M-1\}$, as above,

$$[T_{M,N}^{(2)}(W, Z)]_m = \sum_{n=1}^N \left(\ell_{N,n}^+(h + \theta_{M,m}) W_n, \int_0^{h+\theta_{M,m}} \ell_{N,n}^+(\sigma) d\sigma Z_n \right),$$

while for $m = M$,

$$[T_{M,N}^{(2)}(W, Z)]_M = V^+ P_N^+(W, Z)(h + \theta_{M,M}) = V^+ P_N^+(W, Z)(0) = (0, 0).$$

Thus $T_{M,N}^{(2)}$ is defined as above, but with the last d_X lines of $[T_{M,N}^{(2)}]_{XX}$ and the last d_Y lines of $[T_{M,N}^{(2)}]_{YY}$ substituted with zeros. Finally, if $h < \tau$, for $m \in \{0, \dots, M-1\}$ and $q \in \{1, \dots, Q\}$, and for $m = M$ and $q = Q$,

$$\begin{aligned} [T_{M,N}^{(2)}(W, Z)]_{q,m} &= V^+ P_N^+(W, Z)(h + \theta_{M,m}^{(q)}) \\ &= \begin{cases} \sum_{n=1}^N \left(\ell_{N,n}^+(h + \theta_{M,m}^{(q)}) W_n, \right. \\ \quad \left. \int_0^{h+\theta_{M,m}^{(q)}} \ell_{N,n}^+(\sigma) d\sigma Z_n \right), & q = 1, \\ 0, & q \in \{2, \dots, Q\}. \end{cases} \end{aligned}$$

Then

$$T_{M,N}^{(2)} = \begin{pmatrix} [T_{M,N}^{(2)}]_{XX} & [T_{M,N}^{(2)}]_{XY} \\ [T_{M,N}^{(2)}]_{YX} & [T_{M,N}^{(2)}]_{YY} \end{pmatrix} \in \mathbb{R}^{(d_X+d_Y)(QM+1) \times (d_X+d_Y)N},$$

where

$$\begin{aligned} [T_{M,N}^{(2)}]_{XY} &= \mathbf{0} \in \mathbb{R}^{d_X(QM+1) \times d_Y N}, \\ [T_{M,N}^{(2)}]_{YX} &= \mathbf{0} \in \mathbb{R}^{d_Y(QM+1) \times d_X N}, \end{aligned}$$

and

$$\begin{aligned} [T_{M,N}^{(2)}]_{XX} &\in \mathbb{R}^{d_X(QM+1) \times d_X N}, \\ [T_{M,N}^{(2)}]_{YY} &\in \mathbb{R}^{d_Y(QM+1) \times d_Y N}, \end{aligned}$$

are given by

$$\begin{aligned}
 [T_{M,N}^{(2)}]_{XX} &= \begin{pmatrix} \ell_{N,1}^+(h + \theta_{M,0}^{(1)}) & \cdots & \ell_{N,N}^+(h + \theta_{M,0}^{(1)}) \\ \vdots & \ddots & \vdots \\ \ell_{N,1}^+(h + \theta_{M,M-1}^{(1)}) & \cdots & \ell_{N,N}^+(h + \theta_{M,M-1}^{(1)}) \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \otimes I_{d_X}, \\
 [T_{M,N}^{(2)}]_{YY} &= \begin{pmatrix} \int_0^{h+\theta_{M,0}^{(1)}} \ell_{N,1}^+(\sigma) d\sigma & \cdots & \int_0^{h+\theta_{M,0}^{(1)}} \ell_{N,N}^+(\sigma) d\sigma \\ \vdots & \ddots & \vdots \\ \int_0^{h+\theta_{M,M-1}^{(1)}} \ell_{N,1}^+(\sigma) d\sigma & \cdots & \int_0^{h+\theta_{M,M-1}^{(1)}} \ell_{N,N}^+(\sigma) d\sigma \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \otimes I_{d_Y}.
 \end{aligned}$$

A.1.3 The matrix $U_{M,N}^{(1)}$

Let $(\Phi, \Psi) \in X_M \times Y_M$ and, for $t > 0$, define

$$\kappa(t) := \max_{k \in \{0, \dots, p\}} \{\tau_k < t\}. \quad (\text{A.1})$$

Note that κ is nondecreasing. For $n \in \{1, \dots, N\}$,

$$\begin{aligned}
 [U_{M,N}^{(1)}(\Phi, \Psi)]_n &= \mathcal{F}_s V^- P_M(\Phi, \Psi)(t_{N,n}) \\
 &= \left(\sum_{k=1}^p \int_{-\tau_k}^{-\tau_{k-1}} C_{XX}^{(k)}(s + t_{N,n}, \theta) V_X^- P_M \Phi(t_{N,n} + \theta) d\theta \right. \\
 &\quad + A_{XY}(s + t_{N,n}) V_Y^- P_M \Psi(t_{N,n}) \\
 &\quad + \sum_{k=1}^p B_{XY}^{(k)}(s + t_{N,n}) V_Y^- P_M \Psi(t_{N,n} - \tau_k) \\
 &\quad + \sum_{k=1}^p \int_{-\tau_k}^{-\tau_{k-1}} C_{XY}^{(k)}(s + t_{N,n}, \theta) V_Y^- P_M \Psi(t_{N,n} + \theta) d\theta, \\
 &\quad \sum_{k=1}^p \int_{-\tau_k}^{-\tau_{k-1}} C_{YX}^{(k)}(s + t_{N,n}, \theta) V_X^- P_M \Phi(t_{N,n} + \theta) d\theta \\
 &\quad + A_{YY}(s + t_{N,n}) V_Y^- P_M \Psi(t_{N,n}) \\
 &\quad + \sum_{k=1}^p B_{YY}^{(k)}(s + t_{N,n}) V_Y^- P_M \Psi(t_{N,n} - \tau_k) \\
 &\quad \left. + \sum_{k=1}^p \int_{-\tau_k}^{-\tau_{k-1}} C_{YY}^{(k)}(s + t_{N,n}, \theta) V_Y^- P_M \Psi(t_{N,n} + \theta) d\theta \right).
 \end{aligned}$$

If $h \geq \tau$, define also

$$\hat{N} := \begin{cases} 0, & t_{N,n} > \tau \text{ for all } n \in \{1, \dots, N\}, \\ \max_{n \in \{1, \dots, N\}} \{t_{N,n} \leq \tau\}, & \text{otherwise.} \end{cases}$$

Hence, for $n \in \{1, \dots, \hat{N}\}$ (if $\hat{N} \neq 0$),

$$\begin{aligned}
& [U_{M,N}^{(1)}(\Phi, \Psi)]_n \\
&= \left(\int_{-\tau_{\kappa(t_{N,n})+1}}^{-t_{N,n}} C_{XX}^{(\kappa(t_{N,n})+1)}(s+t_{N,n}, \theta) \sum_{m=0}^M \ell_{M,m}(t_{N,n} + \theta) \Phi_m d\theta \right. \\
&\quad + \sum_{k=\kappa(t_{N,n})+2}^p \int_{-\tau_k}^{-\tau_{k-1}} C_{XX}^{(k)}(s+t_{N,n}, \theta) \sum_{m=0}^M \ell_{M,m}(t_{N,n} + \theta) \Phi_m d\theta \\
&\quad + A_{XY}(s+t_{N,n})\Psi_0 + \sum_{k=1}^{\kappa(t_{N,n})} B_{XY}^{(k)}(s+t_{N,n})\Psi_0 \\
&\quad + \sum_{k=\kappa(t_{N,n})+1}^p B_{XY}^{(k)}(s+t_{N,n}) \sum_{m=0}^M \ell_{M,m}(t_{N,n} - \tau_k) \Psi_m \\
&\quad + \sum_{k=1}^{\kappa(t_{N,n})} \int_{-\tau_k}^{-\tau_{k-1}} C_{XY}^{(k)}(s+t_{N,n}, \theta) \Psi_0 d\theta \\
&\quad + \int_{-t_{N,n}}^{-\tau_{\kappa(t_{N,n})}} C_{XY}^{(\kappa(t_{N,n})+1)}(s+t_{N,n}, \theta) \Psi_0 d\theta \\
&\quad + \int_{-\tau_{\kappa(t_{N,n})+1}}^{-t_{N,n}} C_{XY}^{(\kappa(t_{N,n})+1)}(s+t_{N,n}, \theta) \sum_{m=0}^M \ell_{M,m}(t_{N,n} + \theta) \Psi_m d\theta \\
&\quad + \sum_{k=\kappa(t_{N,n})+2}^p \int_{-\tau_k}^{-\tau_{k-1}} C_{XY}^{(k)}(s+t_{N,n}, \theta) \sum_{m=0}^M \ell_{M,m}(t_{N,n} + \theta) \Psi_m d\theta, \\
&\quad \int_{-\tau_{\kappa(t_{N,n})+1}}^{-t_{N,n}} C_{YX}^{(\kappa(t_{N,n})+1)}(s+t_{N,n}, \theta) \sum_{m=0}^M \ell_{M,m}(t_{N,n} + \theta) \Phi_m d\theta \\
&\quad + \sum_{k=\kappa(t_{N,n})+2}^p \int_{-\tau_k}^{-\tau_{k-1}} C_{YX}^{(k)}(s+t_{N,n}, \theta) \sum_{m=0}^M \ell_{M,m}(t_{N,n} + \theta) \Phi_m d\theta \\
&\quad + A_{YX}(s+t_{N,n})\Psi_0 + \sum_{k=1}^{\kappa(t_{N,n})} B_{YX}^{(k)}(s+t_{N,n})\Psi_0 \\
&\quad + \sum_{k=\kappa(t_{N,n})+1}^p B_{YX}^{(k)}(s+t_{N,n}) \sum_{m=0}^M \ell_{M,m}(t_{N,n} - \tau_k) \Psi_m \\
&\quad + \sum_{k=1}^{\kappa(t_{N,n})} \int_{-\tau_k}^{-\tau_{k-1}} C_{YX}^{(k)}(s+t_{N,n}, \theta) \Psi_0 d\theta \\
&\quad + \int_{-t_{N,n}}^{-\tau_{\kappa(t_{N,n})}} C_{YX}^{(\kappa(t_{N,n})+1)}(s+t_{N,n}, \theta) \Psi_0 d\theta \\
&\quad + \int_{-\tau_{\kappa(t_{N,n})+1}}^{-t_{N,n}} C_{YX}^{(\kappa(t_{N,n})+1)}(s+t_{N,n}, \theta) \sum_{m=0}^M \ell_{M,m}(t_{N,n} + \theta) \Psi_m d\theta \\
&\quad + \sum_{k=\kappa(t_{N,n})+2}^p \int_{-\tau_k}^{-\tau_{k-1}} C_{YX}^{(k)}(s+t_{N,n}, \theta) \sum_{m=0}^M \ell_{M,m}(t_{N,n} + \theta) \Psi_m d\theta \Big)
\end{aligned}$$

(observe that the integral on $[-\tau_{\kappa(t_{N,n})+1}, -t_{N,n}]$ may be zero), and, for $n \in \{\hat{N} + 1, \dots, N\}$,

$$[U_{M,N}^{(1)}(\Phi, \Psi)]_n$$

$$\begin{aligned}
&= \left(A_{XY}(s + t_{N,n})\Psi_0 + \sum_{k=1}^p B_{XY}^{(k)}(s + t_{N,n})\Psi_0 \right. \\
&\quad \left. + \sum_{k=1}^p \int_{-\tau_k}^{-\tau_{k-1}} C_{XY}^{(k)}(s + t_{N,n}, \theta)\Psi_0 d\theta, \right. \\
&\quad \left. A_{YY}(s + t_{N,n})\Psi_0 + \sum_{k=1}^p B_{YY}^{(k)}(s + t_{N,n})\Psi_0 \right. \\
&\quad \left. + \sum_{k=1}^p \int_{-\tau_k}^{-\tau_{k-1}} C_{YY}^{(k)}(s + t_{N,n}, \theta)\Psi_0 d\theta \right).
\end{aligned}$$

For $n \in \{1, \dots, \hat{N}\}$ (if $\hat{N} \neq 0$), let

$$\begin{aligned}
\Gamma_{XY;n,0} &:= A_{XY}(s + t_{N,n}) + \sum_{k=1}^{\kappa(t_{N,n})} B_{XY}^{(k)}(s + t_{N,n}) \\
&\quad + \sum_{k=1}^{\kappa(t_{N,n})} \int_{-\tau_k}^{-\tau_{k-1}} C_{XY}^{(k)}(s + t_{N,n}, \theta) d\theta \\
&\quad + \int_{-t_{N,n}}^{-\tau_{\kappa(t_{N,n})}} C_{XY}^{(\kappa(t_{N,n})+1)}(s + t_{N,n}, \theta) d\theta, \\
\Gamma_{YY;n,0} &:= A_{YY}(s + t_{N,n}) + \sum_{k=1}^{\kappa(t_{N,n})} B_{YY}^{(k)}(s + t_{N,n}) \\
&\quad + \sum_{k=1}^{\kappa(t_{N,n})} \int_{-\tau_k}^{-\tau_{k-1}} C_{YY}^{(k)}(s + t_{N,n}, \theta) d\theta \\
&\quad + \int_{-t_{N,n}}^{-\tau_{\kappa(t_{N,n})}} C_{YY}^{(\kappa(t_{N,n})+1)}(s + t_{N,n}, \theta) d\theta.
\end{aligned}$$

For $n \in \{\hat{N} + 1, \dots, N\}$, let

$$\begin{aligned}
\Gamma_{XY;n,0} &:= A_{XY}(s + t_{N,n}) + \sum_{k=1}^p B_{XY}^{(k)}(s + t_{N,n}) \\
&\quad + \sum_{k=1}^p \int_{-\tau_k}^{-\tau_{k-1}} C_{XY}^{(k)}(s + t_{N,n}, \theta) d\theta, \\
\Gamma_{YY;n,0} &:= A_{YY}(s + t_{N,n}) + \sum_{k=1}^p B_{YY}^{(k)}(s + t_{N,n}) \\
&\quad + \sum_{k=1}^p \int_{-\tau_k}^{-\tau_{k-1}} C_{YY}^{(k)}(s + t_{N,n}, \theta) d\theta.
\end{aligned}$$

Note that $\Gamma_{XY;n,0} \in \mathbb{R}^{d_X \times d_Y}$, $\Gamma_{YY;n,0} \in \mathbb{R}^{d_Y \times d_Y}$. For $m \in \{0, \dots, M\}$ and $n \in \{1, \dots, \hat{N}\}$ (if $\hat{N} \neq 0$), let

$$\begin{aligned}
\Theta_{XX;n,m} &:= \int_{-\tau_{\kappa(t_{N,n})+1}}^{-t_{N,n}} C_{XX}^{(\kappa(t_{N,n})+1)}(s + t_{N,n}, \theta) \ell_{M,m}(t_{N,n} + \theta) d\theta \\
&\quad + \sum_{k=\kappa(t_{N,n})+2}^p \int_{-\tau_k}^{-\tau_{k-1}} C_{XX}^{(k)}(s + t_{N,n}, \theta) \ell_{M,m}(t_{N,n} + \theta) d\theta, \\
\Theta_{XY;n,m} &:= \sum_{k=\kappa(t_{N,n})+1}^p B_{XY}^{(k)}(s + t_{N,n}) \ell_{M,m}(t_{N,n} - \tau_k)
\end{aligned}$$

$$\begin{aligned}
& + \int_{-\tau_{\kappa(t_{N,n})+1}}^{-t_{N,n}} C_{XY}^{(\kappa(t_{N,n})+1)}(s+t_{N,n},\theta) \ell_{M,m}(t_{N,n}+\theta) d\theta \\
& + \sum_{k=\kappa(t_{N,n})+2}^p \int_{-\tau_k}^{-\tau_{k-1}} C_{XY}^{(k)}(s+t_{N,n},\theta) \ell_{M,m}(t_{N,n}+\theta) d\theta, \\
\Theta_{YX;n,m} &:= \int_{-\tau_{\kappa(t_{N,n})+1}}^{-t_{N,n}} C_{YX}^{(\kappa(t_{N,n})+1)}(s+t_{N,n},\theta) \ell_{M,m}(t_{N,n}+\theta) d\theta \\
& + \sum_{k=\kappa(t_{N,n})+2}^p \int_{-\tau_k}^{-\tau_{k-1}} C_{YX}^{(k)}(s+t_{N,n},\theta) \ell_{M,m}(t_{N,n}+\theta) d\theta, \\
\Theta_{YY;n,m} &:= \sum_{k=\kappa(t_{N,n})+1}^p B_{YY}^{(k)}(s+t_{N,n}) \ell_{M,m}(t_{N,n}-\tau_k) \\
& + \int_{-\tau_{\kappa(t_{N,n})+1}}^{-t_{N,n}} C_{YY}^{(\kappa(t_{N,n})+1)}(s+t_{N,n},\theta) \ell_{M,m}(t_{N,n}+\theta) d\theta \\
& + \sum_{k=\kappa(t_{N,n})+2}^p \int_{-\tau_k}^{-\tau_{k-1}} C_{YY}^{(k)}(s+t_{N,n},\theta) \ell_{M,m}(t_{N,n}+\theta) d\theta.
\end{aligned}$$

Note that

$$\begin{aligned}
\Theta_{XX;n,m} &\in \mathbb{R}^{d_X \times d_X}, & \Theta_{XY;n,m} &\in \mathbb{R}^{d_X \times d_Y}, \\
\Theta_{YX;n,m} &\in \mathbb{R}^{d_Y \times d_X}, & \Theta_{YY;n,m} &\in \mathbb{R}^{d_Y \times d_Y}.
\end{aligned}$$

Then

$$U_{M,N}^{(1)} = \begin{pmatrix} [U_{M,N}^{(1)}]_{XX} & [U_{M,N}^{(1)}]_{XY} \\ [U_{M,N}^{(1)}]_{YX} & [U_{M,N}^{(1)}]_{YY} \end{pmatrix} \in \mathbb{R}^{(d_X+d_Y)N \times (d_X+d_Y)(M+1)},$$

where

$$\begin{aligned}
[U_{M,N}^{(1)}]_{XX} &\in \mathbb{R}^{d_X N \times d_X (M+1)}, \\
[U_{M,N}^{(1)}]_{XY} &\in \mathbb{R}^{d_X N \times d_Y (M+1)}, \\
[U_{M,N}^{(1)}]_{YX} &\in \mathbb{R}^{d_Y N \times d_X (M+1)}, \\
[U_{M,N}^{(1)}]_{YY} &\in \mathbb{R}^{d_Y N \times d_Y (M+1)}.
\end{aligned}$$

are given by

$$\begin{aligned}
[U_{M,N}^{(1)}]_{XX} &= \begin{pmatrix} \Theta_{XX;1,0} & \cdots & \Theta_{XX;1,M} \\ \vdots & \ddots & \vdots \\ \Theta_{XX;\hat{N},0} & \cdots & \Theta_{XX;\hat{N},M} \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}, \\
[U_{M,N}^{(1)}]_{XY} &= \begin{pmatrix} \Gamma_{XY;1,0} + \Theta_{XY;1,0} & \Theta_{XY;1,1} & \cdots & \Theta_{XY;1,M} \\ \vdots & \vdots & \ddots & \vdots \\ \Gamma_{XY;\hat{N},0} + \Theta_{XY;\hat{N},0} & \Theta_{XY;\hat{N},1} & \cdots & \Theta_{XY;\hat{N},M} \\ \Gamma_{XY;\hat{N}+1,0} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \Gamma_{XY;N,0} & 0 & \cdots & 0 \end{pmatrix},
\end{aligned}$$

$$\begin{aligned}
[U_{M,N}^{(1)}]_{YX} &= \begin{pmatrix} \Theta_{YX;1,0} & \cdots & \Theta_{YX;1,M} \\ \vdots & \ddots & \vdots \\ \Theta_{YX;\hat{N},0} & \cdots & \Theta_{YX;\hat{N},M} \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}, \\
[U_{M,N}^{(1)}]_{YY} &= \begin{pmatrix} \Gamma_{YY;1,0} + \Theta_{YY;1,0} & \Theta_{YY;1,1} & \cdots & \Theta_{YY;1,M} \\ \vdots & \vdots & \ddots & \vdots \\ \Gamma_{YY;\hat{N},0} + \Theta_{YY;\hat{N},0} & \Theta_{YY;\hat{N},1} & \cdots & \Theta_{YY;\hat{N},M} \\ \Gamma_{YY;\hat{N}+1,0} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \Gamma_{YY;N,0} & 0 & \cdots & 0 \end{pmatrix}.
\end{aligned}$$

Observe that if $\hat{N} = 0$ only the first columns of $[U_{M,N}^{(1)}]_{XY}$ and $[U_{M,N}^{(1)}]_{YY}$ are nonzero.

If $h < \tau$, instead, for $n \in \{1, \dots, N\}$ and $q \in \{0, \dots, Q-1\}$, define $t_{N,n}^{(q)} = qh + t_{N,n}$. Observe that, for $q \in \{1, \dots, Q-1\}$, $t_{N,n} - \tau_k \in (-qh, -(q-1)h]$ if and only if $\kappa(t_{N,n}^{(q-1)}) + 1 \leq k \leq \kappa(t_{N,n}^{(q)})$ and $[t_{N,n} - \tau_k, t_{N,n} - \tau_{k-1}] \cap (-qh, -(q-1)h] \neq \emptyset$ if and only if $\kappa(t_{N,n}^{(q-1)}) + 1 \leq k \leq \kappa(t_{N,n}^{(q)}) + 1$. Moreover, $t_{N,n} - \tau_k \in [-\tau, -(Q-1)h]$ if and only if $k \geq \kappa(t_{N,n}^{(Q-1)}) + 1$ and $[t_{N,n} - \tau_k, t_{N,n} - \tau_{k-1}] \cap [-\tau, -(Q-1)h] \neq \emptyset$ if and only if $k \geq \kappa(t_{N,n}^{(Q-1)}) + 1$. Finally, $t_{N,n} - \tau_k \in (0, h]$ if and only if $k \leq \kappa(t_{N,n})$ and $[t_{N,n} - \tau_k, t_{N,n} - \tau_{k-1}] \cap (0, h] \neq \emptyset$ if and only if $k \leq \kappa(t_{N,n}) + 1$. Observe also that $\kappa(t_{N,n}^{(q-1)})$ and $\kappa(t_{N,n}^{(q)})$ may be equal. For $n \in \{1, \dots, N\}$, $k \in \{1, \dots, p\}$ and $q \in \{1, \dots, Q-1\}$, define $a_{k,0} := \max\{-\tau_k, -t_{N,n}^{(q)}\}$ and

$$\begin{aligned}
a_{k,q} &:= \max\{-\tau_k, -t_{N,n}^{(q)}\}, & a_{k,Q} &:= -\tau_k, \\
b_{k,q} &:= \min\{-\tau_{k-1}, -t_{N,n}^{(q-1)}\}, & b_{k,Q} &:= \min\{-\tau_{k-1}, -t_{N,n}^{(Q-1)}\}, \\
\kappa_{B;n,q} &:= \min\{\kappa(t_{N,n}^{(q)}), p\}, & \kappa_{B;n,Q} &:= p, \\
\kappa_{C;n,q} &:= \min\{\kappa(t_{N,n}^{(q)}) + 1, p\}, & \kappa_{C;n,Q} &:= p.
\end{aligned}$$

Then, for $n \in \{1, \dots, N\}$,

$$\begin{aligned}
& [U_{M,N}^{(1)}(\Phi, \Psi)]_n \\
&= \left(\sum_{q=1}^Q \sum_{k=\kappa(t_{N,n}^{(q-1)})+1}^{\kappa_{C;n,q}} \int_{a_{k,q}}^{b_{k,q}} C_{XX}^{(k)}(s+t_{N,n}, \theta) \sum_{m=0}^M \ell_{M,m}^{(q)}(t_{N,n} + \theta) \Phi_m^{(q)} d\theta \right. \\
&\quad + A_{XY}(s+t_{N,n}) \Psi_0^{(1)} + \sum_{k=1}^{\kappa(t_{N,n})} B_{XY}^{(k)}(s+t_{N,n}) \Psi_0^{(1)} \\
&\quad + \sum_{q=1}^Q \sum_{k=\kappa(t_{N,n}^{(q-1)})+1}^{\kappa_{B;n,q}} B_{XY}^{(k)}(s+t_{N,n}) \sum_{m=0}^M \ell_{M,m}^{(q)}(t_{N,n} - \tau_k) \Psi_m^{(q)} \\
&\quad + \sum_{k=1}^{\kappa(t_{N,n})+1} \int_{a_{k,0}}^{-\tau_{k-1}} C_{XY}^{(k)}(s+t_{N,n}, \theta) \Psi_0^{(1)} d\theta \\
&\quad + \sum_{q=1}^Q \sum_{k=\kappa(t_{N,n}^{(q-1)})+1}^{\kappa_{C;n,q}} \int_{a_{k,q}}^{b_{k,q}} C_{XY}^{(k)}(s+t_{N,n}, \theta) \sum_{m=0}^M \ell_{M,m}^{(q)}(t_{N,n} + \theta) \Psi_m^{(q)} d\theta, \\
&\quad \sum_{q=1}^Q \sum_{k=\kappa(t_{N,n}^{(q-1)})+1}^{\kappa_{C;n,q}} \int_{a_{k,q}}^{b_{k,q}} C_{YX}^{(k)}(s+t_{N,n}, \theta) \sum_{m=0}^M \ell_{M,m}^{(q)}(t_{N,n} + \theta) \Phi_m^{(q)} d\theta \\
&\quad + A_{YY}(s+t_{N,n}) \Psi_0^{(1)} + \sum_{k=1}^{\kappa(t_{N,n})} B_{YY}^{(k)}(s+t_{N,n}) \Psi_0^{(1)} \\
&\quad + \sum_{q=1}^Q \sum_{k=\kappa(t_{N,n}^{(q-1)})+1}^{\kappa_{B;n,q}} B_{YY}^{(k)}(s+t_{N,n}) \sum_{m=0}^M \ell_{M,m}^{(q)}(t_{N,n} - \tau_k) \Psi_m^{(q)} \\
&\quad + \sum_{k=1}^{\kappa(t_{N,n})+1} \int_{a_{k,0}}^{-\tau_{k-1}} C_{YY}^{(k)}(s+t_{N,n}, \theta) \Psi_0^{(1)} d\theta, \\
&\quad \left. + \sum_{q=1}^Q \sum_{k=\kappa(t_{N,n}^{(q-1)})+1}^{\kappa_{C;n,q}} \int_{a_{k,q}}^{b_{k,q}} C_{YY}^{(k)}(s+t_{N,n}, \theta) \sum_{m=0}^M \ell_{M,m}^{(q)}(t_{N,n} + \theta) \Psi_m^{(q)} d\theta \right).
\end{aligned}$$

with the convention that $\sum_{k=k_1}^{k_2} a_k = 0$ if $k_2 < k_1$. Observe that some of the integrals may be zero. For $n \in \{1, \dots, N\}$, let

$$\begin{aligned}
\Gamma_{XY;n,0}^{(1)} &:= A_{XY}(s+t_{N,n}) + \sum_{k=1}^{\kappa(t_{N,n})} B_{XY}^{(k)}(s+t_{N,n}) \\
&\quad + \sum_{k=1}^{\kappa(t_{N,n})+1} \int_{a_{k,0}}^{-\tau_{k-1}} C_{XY}^{(k)}(s+t_{N,n}, \theta) d\theta, \\
\Gamma_{YY;n,0}^{(1)} &:= A_{YY}(s+t_{N,n}) + \sum_{k=1}^{\kappa(t_{N,n})} B_{YY}^{(k)}(s+t_{N,n}) \\
&\quad + \sum_{k=1}^{\kappa(t_{N,n})+1} \int_{a_{k,0}}^{-\tau_{k-1}} C_{YY}^{(k)}(s+t_{N,n}, \theta) d\theta.
\end{aligned}$$

Note that $\Gamma_{XY;n,0}^{(1)} \in \mathbb{R}^{d_X \times d_Y}$, $\Gamma_{YY;n,0}^{(1)} \in \mathbb{R}^{d_Y \times d_Y}$. For $n \in \{1, \dots, N\}$, $m \in \{0, \dots, M\}$ and $q \in \{1, \dots, Q\}$, define

$$\begin{aligned}\Theta_{XX;n,m}^{(q)} &:= \sum_{k=\kappa(t_{N,n}^{(q-1)})+1}^{\kappa_{C;n,q}} \int_{a_{k,q}}^{b_{k,q}} C_{XX}^{(k)}(s+t_{N,n}, \theta) \ell_{M,m}^{(q)}(t_{N,n} + \theta) d\theta, \\ \Theta_{XY;n,m}^{(q)} &:= \sum_{k=\kappa(t_{N,n}^{(q-1)})+1}^{\kappa_{B;n,q}} B_{XY}^{(k)}(s+t_{N,n}) \ell_{M,m}^{(q)}(t_{N,n} - \tau_k) \\ &\quad + \sum_{k=\kappa(t_{N,n}^{(q-1)})+1}^{\kappa_{C;n,q}} \int_{a_{k,q}}^{b_{k,q}} C_{XY}^{(k)}(s+t_{N,n}, \theta) \ell_{M,m}^{(q)}(t_{N,n} + \theta) d\theta, \\ \Theta_{YX;n,m}^{(q)} &:= \sum_{k=\kappa(t_{N,n}^{(q-1)})+1}^{\kappa_{C;n,q}} \int_{a_{k,q}}^{b_{k,q}} C_{YX}^{(k)}(s+t_{N,n}, \theta) \ell_{M,m}^{(q)}(t_{N,n} + \theta) d\theta, \\ \Theta_{YY;n,m}^{(q)} &:= \sum_{k=\kappa(t_{N,n}^{(q-1)})+1}^{\kappa_{B;n,q}} B_{YY}^{(k)}(s+t_{N,n}) \ell_{M,m}^{(q)}(t_{N,n} - \tau_k) \\ &\quad + \sum_{k=\kappa(t_{N,n}^{(q-1)})+1}^{\kappa_{C;n,q}} \int_{a_{k,q}}^{b_{k,q}} C_{YY}^{(k)}(s+t_{N,n}, \theta) \ell_{M,m}^{(q)}(t_{N,n} + \theta) d\theta.\end{aligned}$$

Note that

$$\begin{aligned}\Theta_{XX;n,m}^{(q)} &\in \mathbb{R}^{d_X \times d_X}, & \Theta_{XY;n,m}^{(q)} &\in \mathbb{R}^{d_X \times d_Y}, \\ \Theta_{YX;n,m}^{(q)} &\in \mathbb{R}^{d_Y \times d_X}, & \Theta_{YY;n,m}^{(q)} &\in \mathbb{R}^{d_Y \times d_Y}\end{aligned}$$

and recall that, for $q \in \{1, \dots, Q-1\}$, $\Phi_M^{(q)} = \Phi_0^{(q+1)}$ and $\Psi_M^{(q)} = \Psi_0^{(q+1)}$. Then

$$U_{M,N}^{(1)} = \begin{pmatrix} [U_{M,N}^{(1)}]_{XX} & [U_{M,N}^{(1)}]_{XY} \\ [U_{M,N}^{(1)}]_{YX} & [U_{M,N}^{(1)}]_{YY} \end{pmatrix} \in \mathbb{R}^{(d_X+d_Y)N \times (d_X+d_Y)(QM+1)},$$

where $[U_{M,N}^{(1)}]_{XX} \in \mathbb{R}^{d_X N \times d_X (QM+1)}$, $[U_{M,N}^{(1)}]_{XY} \in \mathbb{R}^{d_X N \times d_Y (QM+1)}$, $[U_{M,N}^{(1)}]_{YX} \in \mathbb{R}^{d_Y N \times d_X (QM+1)}$ and $[U_{M,N}^{(1)}]_{YY} \in \mathbb{R}^{d_Y N \times d_Y (QM+1)}$ are given by

$$\begin{aligned}[U_{M,N}^{(1)}]_{XX} &= \begin{pmatrix} \Theta_{XX;1,0}^{(1)} & \cdots & \Theta_{XX;1,M-1}^{(1)} & \Theta_{XX;1,M}^{(1)} + \Theta_{XX;1,0}^{(2)} & \Theta_{XX;1,1}^{(2)} & \cdots & \Theta_{XX;1,M-1}^{(2)} & \cdots \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \cdots \\ \Theta_{XX;N,0}^{(1)} & \cdots & \Theta_{XX;N,M-1}^{(1)} & \Theta_{XX;N,M}^{(1)} + \Theta_{XX;N,0}^{(2)} & \Theta_{XX;N,1}^{(2)} & \cdots & \Theta_{XX;N,M-1}^{(2)} & \cdots \\ & & & \Theta_{XX;1,M}^{(Q-1)} + \Theta_{XX;1,0}^{(Q)} & \Theta_{XX;1,1}^{(Q)} & \cdots & \Theta_{XX;1,M-1}^{(Q)} & \Theta_{XX;1,M}^{(Q)} \\ & & & \vdots & \vdots & \ddots & \vdots & \vdots \\ & & & \Theta_{XX;N,M}^{(Q-1)} + \Theta_{XX;N,0}^{(Q)} & \Theta_{XX;N,1}^{(Q)} & \cdots & \Theta_{XX;N,M-1}^{(Q)} & \Theta_{XX;N,M}^{(Q)} \end{pmatrix}, \\ [U_{M,N}^{(1)}]_{XY} &= \begin{pmatrix} \Gamma_{XY;1,0}^{(1)} + \Theta_{XY;1,0}^{(1)} & \cdots & \Theta_{XY;1,M-1}^{(1)} & \Theta_{XY;1,M}^{(1)} + \Theta_{XY;1,0}^{(2)} & \Theta_{XY;1,1}^{(2)} & \cdots & \Theta_{XY;1,M-1}^{(2)} & \cdots \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \cdots \\ \Gamma_{XY;N,0}^{(1)} + \Theta_{XY;N,0}^{(1)} & \cdots & \Theta_{XY;N,M-1}^{(1)} & \Theta_{XY;N,M}^{(1)} + \Theta_{XY;N,0}^{(2)} & \Theta_{XY;N,1}^{(2)} & \cdots & \Theta_{XY;N,M-1}^{(2)} & \cdots \\ & & & \Theta_{XY;1,M}^{(Q-1)} + \Theta_{XY;1,0}^{(Q)} & \Theta_{XY;1,1}^{(Q)} & \cdots & \Theta_{XY;1,M-1}^{(Q)} & \Theta_{XY;1,M}^{(Q)} \\ & & & \vdots & \vdots & \ddots & \vdots & \vdots \\ & & & \Theta_{XY;N,M}^{(Q-1)} + \Theta_{XY;N,0}^{(Q)} & \Theta_{XY;N,1}^{(Q)} & \cdots & \Theta_{XY;N,M-1}^{(Q)} & \Theta_{XY;N,M}^{(Q)} \end{pmatrix}, \\ [U_{M,N}^{(1)}]_{YX} &= \begin{pmatrix} \Theta_{YX;1,0}^{(1)} & \cdots & \Theta_{YX;1,M-1}^{(1)} & \Theta_{YX;1,M}^{(1)} + \Theta_{YX;1,0}^{(2)} & \Theta_{YX;1,1}^{(2)} & \cdots & \Theta_{YX;1,M-1}^{(2)} & \cdots \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \cdots \\ \Theta_{YX;N,0}^{(1)} & \cdots & \Theta_{YX;N,M-1}^{(1)} & \Theta_{YX;N,M}^{(1)} + \Theta_{YX;N,0}^{(2)} & \Theta_{YX;N,1}^{(2)} & \cdots & \Theta_{YX;N,M-1}^{(2)} & \cdots \\ & & & \Theta_{YX;1,M}^{(Q-1)} + \Theta_{YX;1,0}^{(Q)} & \Theta_{YX;1,1}^{(Q)} & \cdots & \Theta_{YX;1,M-1}^{(Q)} & \Theta_{YX;1,M}^{(Q)} \\ & & & \vdots & \vdots & \ddots & \vdots & \vdots \\ & & & \Theta_{YX;N,M}^{(Q-1)} + \Theta_{YX;N,0}^{(Q)} & \Theta_{YX;N,1}^{(Q)} & \cdots & \Theta_{YX;N,M-1}^{(Q)} & \Theta_{YX;N,M}^{(Q)} \end{pmatrix},\end{aligned}$$

$$[U_{M,N}^{(1)}]_{YY} = \begin{pmatrix} \Gamma_{YY;1,0}^{(1)} + \Theta_{YY;1,0}^{(1)} & \cdots & \Theta_{YY;1,M-1}^{(1)} & \Theta_{YY;1,M}^{(1)} + \Theta_{YY;1,0}^{(2)} & \Theta_{YY;1,1}^{(2)} & \cdots & \Theta_{YY;1,M-1}^{(2)} & \cdots \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \cdots \\ \Gamma_{YY;N,0}^{(1)} + \Theta_{YY;N,0}^{(1)} & \cdots & \Theta_{YY;N,M-1}^{(1)} & \Theta_{YY;N,M}^{(1)} + \Theta_{YY;N,0}^{(2)} & \Theta_{YY;N,1}^{(2)} & \cdots & \Theta_{YY;N,M-1}^{(2)} & \cdots \\ & & & \Theta_{YY;1,M}^{(Q-1)} + \Theta_{YY;1,0}^{(Q)} & \Theta_{YY;1,1}^{(Q)} & \cdots & \Theta_{YY;1,M-1}^{(Q)} & \Theta_{YY;1,M}^{(Q)} \\ & & & \vdots & \vdots & \ddots & \vdots & \vdots \\ \cdots & & & \Theta_{YY;N,M}^{(Q-1)} + \Theta_{YY;N,0}^{(Q)} & \Theta_{YY;N,1}^{(Q)} & \cdots & \Theta_{YY;N,M-1}^{(Q)} & \Theta_{YY;N,M}^{(Q)} \end{pmatrix}.$$

A.1.4 The matrix $U_N^{(2)}$

Let $(W, Z) \in X_N^+ \times Y_N^+$. Define $\kappa(t)$ as in (A.1), for $t > 0$, and recall that $t_{N,n} - \tau_k \in (0, h]$ if and only if $k \leq \kappa(t_{N,n})$ and $[t_{N,n} - \tau_k, t_{N,n} - \tau_{k-1}] \cap (0, h] \neq \emptyset$ if and only if $k \leq \kappa(t_{N,n}) + 1$. For $n \in \{1, \dots, N\}$ define $a_n := \max\{-\tau, -t_{N,n}\}$. For $n \in \{1, \dots, N\}$,

$$\begin{aligned} & [U_N^{(2)}(W, Z)]_n \\ &= \mathcal{F}_s V^+ P_N^+(W, Z)(t_{N,n}) \\ &= \left(\sum_{k=1}^p \int_{-\tau_k}^{-\tau_{k-1}} C_{XX}^{(k)}(s + t_{N,n}, \theta) V_X^+ P_N^+ W(t_{N,n} + \theta) d\theta \right. \\ &\quad + A_{XY}(s + t_{N,n}) V_Y^+ P_N^+ Z(t_{N,n}) \\ &\quad + \sum_{k=1}^p B_{XY}^{(k)}(s + t_{N,n}) V_Y^+ P_N^+ Z(t_{N,n} - \tau_k) \\ &\quad + \sum_{k=1}^p \int_{-\tau_k}^{-\tau_{k-1}} C_{XY}^{(k)}(s + t_{N,n}, \theta) V_Y^+ P_N^+ Z(t + \theta) d\theta, \\ &\quad \left. \sum_{k=1}^p \int_{-\tau_k}^{-\tau_{k-1}} C_{YX}^{(k)}(s + t_{N,n}, \theta) V_X^+ P_N^+ W(t_{N,n} + \theta) d\theta \right. \\ &\quad + A_{YY}(s + t_{N,n}) V_Y^+ P_N^+ Z(t_{N,n}) \\ &\quad + \sum_{k=1}^p B_{YY}^{(k)}(s + t_{N,n}) V_Y^+ P_N^+ Z(t_{N,n} - \tau_k) \\ &\quad + \left. \sum_{k=1}^p \int_{-\tau_k}^{-\tau_{k-1}} C_{YY}^{(k)}(s + t_{N,n}, \theta) V_Y^+ P_N^+ Z(t_{N,n} + \theta) d\theta \right) \\ &= \left(\sum_{k=1}^{\kappa(t_{N,n})} \int_{-\tau_k}^{-\tau_{k-1}} C_{XX}^{(k)}(s + t_{N,n}, \theta) \sum_{i=1}^N \ell_{N,i}^+(t_{N,n} + \theta) W_i d\theta \right. \\ &\quad + \int_{a_n}^{-\tau_{\kappa(t_{N,n})}} C_{XX}^{(\kappa(t_{N,n})+1)}(s + t_{N,n}, \theta) \sum_{i=1}^N \ell_{N,i}^+(t_{N,n} + \theta) W_i d\theta \\ &\quad + A_{XY}(s + t_{N,n}) \int_0^{t_{N,n}} \sum_{i=1}^N \ell_{N,i}^+(\sigma) Z_i d\sigma \\ &\quad + \sum_{k=1}^{\kappa(t_{N,n})} B_{XY}^{(k)}(s + t_{N,n}) \int_0^{t_{N,n} - \tau_k} \sum_{i=1}^N \ell_{N,i}^+(\sigma) Z_i d\sigma \\ &\quad + \sum_{k=1}^{\kappa(t_{N,n})} \int_{-\tau_k}^{-\tau_{k-1}} C_{XY}^{(k)}(s + t_{N,n}, \theta) \int_0^{t_{N,n} + \theta} \sum_{i=1}^N \ell_{N,i}^+(\sigma) Z_i d\sigma d\theta, \\ &\quad + \left. \int_{a_n}^{-\tau_{\kappa(t_{N,n})}} C_{XY}^{(\kappa(t_{N,n})+1)}(s + t_{N,n}, \theta) \int_0^{t_{N,n} + \theta} \sum_{i=1}^N \ell_{N,i}^+(\sigma) Z_i d\sigma d\theta, \right. \end{aligned}$$

$$\begin{aligned}
& \sum_{k=1}^{\kappa(t_{N,n})} \int_{-\tau_k}^{-\tau_{k-1}} C_{YX}^{(k)}(s+t_{N,n}, \theta) \sum_{i=1}^N \ell_{N,i}^+(t_{N,n} + \theta) W_i \, d\theta \\
& + \int_{a_n}^{-\tau_{\kappa(t_{N,n})}} C_{YX}^{(\kappa(t_{N,n})+1)}(s+t_{N,n}, \theta) \sum_{i=1}^N \ell_{N,i}^+(t_{N,n} + \theta) W_i \, d\theta \\
& + A_{YY}(s+t_{N,n}) \int_0^{t_{N,n}} \sum_{i=1}^N \ell_{N,i}^+(\sigma) Z_i \, d\sigma \\
& + \sum_{k=1}^{\kappa(t_{N,n})} B_{YY}^{(k)}(s+t_{N,n}) \int_0^{t_{N,n}-\tau_k} \sum_{i=1}^N \ell_{N,i}^+(\sigma) Z_i \, d\sigma \\
& + \sum_{k=1}^{\kappa(t_{N,n})} \int_{-\tau_k}^{-\tau_{k-1}} C_{YY}^{(k)}(s+t_{N,n}, \theta) \int_0^{t_{N,n}+\theta} \sum_{i=1}^N \ell_{N,i}^+(\sigma) Z_i \, d\sigma \, d\theta, \\
& + \int_{a_n}^{-\tau_{\kappa(t_{N,n})}} C_{YY}^{(\kappa(t_{N,n})+1)}(s+t_{N,n}, \theta) \int_0^{t_{N,n}+\theta} \sum_{i=1}^N \ell_{N,i}^+(\sigma) Z_i \, d\sigma \, d\theta,
\end{aligned}$$

with the convention that $\sum_{k=k_1}^{k_2} a_k = 0$ if $k_2 < k_1$. Observe that $C_{XX}^{(\kappa(t_{N,n})+1)}$ and the analogous terms are undefined if $\kappa(t_{N,n}) = p$: in that case, however, $a_n = -\tau_{\kappa(t_{N,n})} = -\tau$, hence the corresponding integrals are zero anyway. For $n \in \{1, \dots, N\}$ and $i \in \{1, \dots, N\}$, let

$$\begin{aligned}
\Lambda_{XX;n,i} &:= \sum_{k=1}^{\kappa(t_{N,n})} \int_{-\tau_k}^{-\tau_{k-1}} C_{XX}^{(k)}(s+t_{N,n}, \theta) \ell_{N,i}^+(t_{N,n} + \theta) \, d\theta \\
& + \int_{a_n}^{-\tau_{\kappa(t_{N,n})}} C_{XX}^{(\kappa(t_{N,n})+1)}(s+t_{N,n}, \theta) \ell_{N,i}^+(t_{N,n} + \theta) \, d\theta \\
\Lambda_{XY;n,i} &:= A_{XY}(s+t_{N,n}) \int_0^{t_{N,n}} \ell_{N,i}^+(\sigma) \, d\sigma \\
& + \sum_{k=1}^{\kappa(t_{N,n})} B_{XY}^{(k)}(s+t_{N,n}) \int_0^{t_{N,n}-\tau_k} \ell_{N,i}^+(\sigma) \, d\sigma \\
& + \sum_{k=1}^{\kappa(t_{N,n})} \int_{-\tau_k}^{-\tau_{k-1}} C_{XY}^{(k)}(s+t_{N,n}, \theta) \int_0^{t_{N,n}+\theta} \ell_{N,i}^+(\sigma) \, d\sigma \, d\theta, \\
& + \int_{a_n}^{-\tau_{\kappa(t_{N,n})}} C_{XY}^{(\kappa(t_{N,n})+1)}(s+t_{N,n}, \theta) \int_0^{t_{N,n}+\theta} \ell_{N,i}^+(\sigma) \, d\sigma \, d\theta, \\
\Lambda_{YX;n,i} &:= \sum_{k=1}^{\kappa(t_{N,n})} \int_{-\tau_k}^{-\tau_{k-1}} C_{YX}^{(k)}(s+t_{N,n}, \theta) \ell_{N,i}^+(t_{N,n} + \theta) \, d\theta \\
& + \int_{a_n}^{-\tau_{\kappa(t_{N,n})}} C_{YX}^{(\kappa(t_{N,n})+1)}(s+t_{N,n}, \theta) \ell_{N,i}^+(t_{N,n} + \theta) \, d\theta \\
\Lambda_{YY;n,i} &:= A_{YY}(s+t_{N,n}) \int_0^{t_{N,n}} \ell_{N,i}^+(\sigma) \, d\sigma \\
& + \sum_{k=1}^{\kappa(t_{N,n})} B_{YY}^{(k)}(s+t_{N,n}) \int_0^{t_{N,n}-\tau_k} \ell_{N,i}^+(\sigma) \, d\sigma \\
& + \sum_{k=1}^{\kappa(t_{N,n})} \int_{-\tau_k}^{-\tau_{k-1}} C_{YY}^{(k)}(s+t_{N,n}, \theta) \int_0^{t_{N,n}+\theta} \ell_{N,i}^+(\sigma) \, d\sigma \, d\theta, \\
& + \int_{a_n}^{-\tau_{\kappa(t_{N,n})}} C_{YY}^{(\kappa(t_{N,n})+1)}(s+t_{N,n}, \theta) \int_0^{t_{N,n}+\theta} \ell_{N,i}^+(\sigma) \, d\sigma \, d\theta.
\end{aligned}$$

Note that

$$\begin{aligned}\Lambda_{XX;n,i} &\in \mathbb{R}^{d_X \times d_X}, & \Lambda_{XY;n,i} &\in \mathbb{R}^{d_X \times d_Y}, \\ \Lambda_{YX;n,i} &\in \mathbb{R}^{d_Y \times d_X}, & \Lambda_{YY;n,i} &\in \mathbb{R}^{d_Y \times d_Y}.\end{aligned}$$

Then

$$U_N^{(2)} = \begin{pmatrix} [U_N^{(2)}]_{XX} & [U_N^{(2)}]_{XY} \\ [U_N^{(2)}]_{YX} & [U_N^{(2)}]_{YY} \end{pmatrix} \in \mathbb{R}^{(d_X+d_Y)N \times (d_X+d_Y)N},$$

where

$$\begin{aligned}[U_N^{(2)}]_{XX} &= \begin{pmatrix} \Lambda_{XX;1,1} & \cdots & \Lambda_{XX;1,N} \\ \vdots & \ddots & \vdots \\ \Lambda_{XX;N,1} & \cdots & \Lambda_{XX;N,N} \end{pmatrix} \in \mathbb{R}^{d_X N \times d_X N}, \\ [U_N^{(2)}]_{XY} &= \begin{pmatrix} \Lambda_{XY;1,1} & \cdots & \Lambda_{XY;1,N} \\ \vdots & \ddots & \vdots \\ \Lambda_{XY;N,1} & \cdots & \Lambda_{XY;N,N} \end{pmatrix} \in \mathbb{R}^{d_X N \times d_Y N}, \\ [U_N^{(2)}]_{YX} &= \begin{pmatrix} \Lambda_{YX;1,1} & \cdots & \Lambda_{YX;1,N} \\ \vdots & \ddots & \vdots \\ \Lambda_{YX;N,1} & \cdots & \Lambda_{YX;N,N} \end{pmatrix} \in \mathbb{R}^{d_Y N \times d_X N}, \\ [U_N^{(2)}]_{YY} &= \begin{pmatrix} \Lambda_{YY;1,1} & \cdots & \Lambda_{YY;1,N} \\ \vdots & \ddots & \vdots \\ \Lambda_{YY;N,1} & \cdots & \Lambda_{YY;N,N} \end{pmatrix} \in \mathbb{R}^{d_Y N \times d_Y N}.\end{aligned}$$

A.2 NUMERICAL CHOICES IN THE IMPLEMENTATION

A.2.1 Discretization nodes

The discretization of the function spaces is based on interpolation on Chebyshev zeros (for functions on $[0, h]$) and Chebyshev extrema (for functions on $[-\tau, 0]$), which are, respectively, the zeros and extremal points of Chebyshev polynomials of the first kind (see [subsection 2.3.3](#)). Recall that for the spaces of functions on $[0, h]$ this is required by [hypothesis \(H6.1\)](#) and observe that Chebyshev extrema fulfill the requirement of including the endpoints of the interval of definition among the nodes, according to [subsection 4.3.1](#). Moreover, both Chebyshev zeros and extrema are advantageous for their optimal properties, as described in [subsection 2.3.3](#).

Recall the changes of variable from $[-1, 1]$ to $[a, b]$ and vice versa defined in [\(2.3\)](#) and [\(2.4\)](#).

In accordance with [hypothesis \(H6.1\)](#), for $N \in \mathbb{N} \setminus \{0\}$ the nodes of the mesh $\Omega_N^+ := \{t_{N,n}\}_{n \in \{1, \dots, N\}}$ on $[0, h]$ are defined as

$$t_{N,n} := \frac{h}{2}(1 - x_{N,n}), \quad n \in \{1, \dots, N\}.$$

Observe that the nodes are sorted from left to right.

As for $[-\tau, 0]$, if $h \geq \tau$, for $M \in \mathbb{N} \setminus \{0\}$ the nodes of the mesh $\Omega_M := \{\theta_{M,m}\}_{m \in \{0, \dots, M\}}$ are defined as

$$\theta_{M,m} := \frac{\tau}{2}(y_{N,n} - 1), \quad m \in \{0, \dots, M\}.$$

If instead $h < \tau$, let $M \in \mathbb{N} \setminus \{0\}$ and recall from [subsection 4.3.1](#) that Q is the minimum positive integer q such that $qh \geq \tau$, and that $\theta^{(q)} := -qh$ for $q \in \{0, \dots, Q-1\}$ and $\theta^{(Q)} := -\tau$. For $q \in \{1, \dots, Q-1\}$ the nodes of the mesh $\Omega_M^{(q)} := \{\theta_{M,m}^{(q)}\}_{m \in \{0, \dots, M\}}$ are defined as

$$\theta_{M,m}^{(q)} := h \left(\frac{y_{M,m} + 1}{2} - q \right), \quad m \in \{0, \dots, M\},$$

while the nodes of the mesh $\Omega_M^{(Q)} := \{\theta_{M,m}^{(Q)}\}_{m \in \{0, \dots, M\}}$ are defined as

$$\theta_{M,m}^{(Q)} := \frac{\tau - (Q-1)h}{2} y_{M,m} - \frac{\tau + (Q-1)h}{2}, \quad m \in \{0, \dots, M\}.$$

A.2.2 Barycentric formula for Lagrange interpolation

The coefficients of the matrices in [section A.1](#) contain evaluations of Lagrange coefficients (recall [subsection 2.3.2](#)). In the current implementation we resorted to compute them by barycentric interpolation [\[8\]](#), observing that $\ell_n(x_i) = \delta_{n,i}$ for $n, i \in \{1, \dots, N\}$ (with $\delta_{n,i}$ the Kronecker delta). This choice is motivated by the excellent properties of barycentric interpolation in terms of efficiency and stability.

By defining the nodal polynomial π as

$$\ell(x) := \prod_{i=1}^n (x - x_i)$$

and the barycentric weights $\{w_i\}_{i \in \{1, \dots, n\}}$ as

$$w_i(x) := \frac{1}{\prod_{j \neq i} (x_i - x_j)}, \quad i \in \{1, \dots, n\},$$

it is possible to rewrite the interpolating polynomial as

$$p(x) = \ell(x) \sum_{i=1}^n \frac{w_i}{x - x_i} f_i, \quad (\text{A.2})$$

which is called the first form of the barycentric interpolation formula [\[8\]](#). This formula constitutes an improvement in computational cost with respect to the Lagrange formula. First, once the quantities $\{w_i\}$, which are independent of the interpolated values $\{f_i\}$ and cost $O(n^2)$, have been computed for the chosen nodes, evaluations of p cost $O(n)$ arithmetic operations, instead of $O(n^2)$. Second, updating the formula by adding another interpolation node also costs $O(n)$ operations.

By observing that

$$1 = \ell(x) \sum_{i=1}^n \frac{w_i}{x - x_i} \quad (\text{A.3})$$

and dividing (A.2) by (A.3) we obtain

$$p(x) = \frac{\sum_{i=1}^n \frac{w_i}{x - x_i} f_i}{\sum_{i=1}^n \frac{w_i}{x - x_i}},$$

which is called the second (true) form of the barycentric interpolation formula [8]. This formula has the additional advantage that the weights $\{w_i\}$ appear both in the numerator and the denominator, so that any factor common to all weights can be canceled. Moreover, in case $x \approx x_i \neq x$ for some $i \in \{1, \dots, n\}$, where the term $\frac{w_i}{x - x_i}$ would be very large, again the same numbers appear both in the numerator and the denominator and the formula remains numerically stable.

For Chebyshev zeros and extrema, explicit expressions for the barycentric weights are available, namely for Chebyshev zeros

$$w_i = (-1)^i \sin \frac{(2i+1)\pi}{2n+2}, \quad i \in \{1, \dots, n\},$$

and for Chebyshev extrema (with the caveat that nodes are numbered from 0 to n)

$$w_i = \begin{cases} (-1)^{i\frac{1}{2}} & \text{if } i = 0 \text{ or } i = n, \\ (-1)^i & \text{if } i \in \{1, \dots, n-1\}. \end{cases}$$

These expressions are valid for Chebyshev nodes on any interval, since changing the interval requires only to scale the weights by multiplication and, as already observed, common factors are canceled.

More details and references on barycentric formulas may be found in the cited paper [8].

A.2.3 Clenshaw–Curtis quadrature

In Remark 4.15 we observe that it may not be possible to compute the integrals appearing in $\mathcal{F}_s$ exactly. In order to approximate them we choose the Clenshaw–Curtis quadrature formula [95], an interpolatory quadrature formula based on Chebyshev extrema. This choice is motivated by the fact that Clenshaw–Curtis quadrature achieves an infinite order of convergence for smooth integrands, thus preserving the convergence order proved in Theorem 6.13.

Considering the function f defined on $[-1, 1]$, for $N \in \mathbb{N} \setminus \{0\}$, the integral of f over $[-1, 1]$ is approximated by

$$\sum_{n=0}^N w_n f(y_{N,n}),$$

where $\{y_{N,n}\}_{n \in \{0, \dots, N\}}$ are Chebyshev extrema on $[-1, 1]$ as defined in (2.2) and $\{w_n\}_{n \in \{0, \dots, N\}}$ are quadrature weights which can be computed explicitly as described in [95, chapter 12, program clencurt.m]. For each N , the weights are all positive and their sum is 2.

Recalling the change of variable (2.3), the integral of a function f on $[a, b]$ can be written as

$$\int_a^b f(\sigma) \, d\sigma = \frac{b-a}{2} \int_{-1}^1 f(\sigma(s)) \, ds$$

and it is approximated by

$$\frac{b-a}{2} \sum_{n=0}^N w_n f(\sigma(y_{N,n})).$$

More details and references on Clenshaw–Curtis quadrature can be found in [95–97].

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ERRATA CORRIGE

Latest update: 28 October 2019

After submitting this thesis for the final evaluation on 25 January 2018, I made some corrections directly in the text, due to some typographical errors in punctuation and to the following mistakes:

- page 23, line 1: “ $0 < \eta < \bar{\eta}$ ” was previously “ $0 < |\eta| < \bar{\eta}$ ”;
- page 37, Theorem 3.7: “ $u(t) \in \mathcal{M}_{\lambda,t}$ ” was previously “ $u(t) \in \mathcal{M}_{\lambda,0}$ ”;
- page 43, line before (3.14): “around $\bar{x}$ ” was previously “around $\bar{y}$ ”;
- page 81, line 10 of section 6.2: reference to (6.3) was previously (5.3).
- page 89, line 12 of subsection 6.4.3: reference to Proposition 6.12 and Theorem 6.13 was previously Proposition 5.15 and Theorem 5.16.
- page 96, statement and proof of Proposition 7.4: “ $\mathcal{L}_N$ ” was previously “ $\mathcal{L}_N^+$ ” (except for the first instance).

Moreover, the following corrections should be made:

- page 64, before Theorem 5.1: add “(Here and in the following, consider K and C to be prolonged by 0 where they are not defined.)”;
- page 64, Theorem 5.1: change “If the interval...” to “If

$$\operatorname{ess\,sup}_{\sigma \in [0,\tau]} \int_0^\tau |K(r,\sigma)| \, dr < +\infty$$

and the interval...”;

- page 64, Corollary 5.2: change “If the interval...” to “If

$$\operatorname{ess\,sup}_{\sigma \in [0,\tau]} \int_0^\tau |C(s+r,\sigma-r)| \, dr < +\infty \quad (5.5 \text{ bis})$$

and the interval...”;

- page 67, Corollary 5.7: change “If the interval...” to “If (5.5 bis) holds and the interval...”;
- page 69, hypothesis (H5.2): change “the hypothesis of Corollary 5.2 holds” to “the hypotheses of Corollary 5.2 hold”;
- page 69, line 6 below hypothesis (H5.4): change “Indeed, the interval...” to “Indeed,

$$\operatorname{ess\,sup}_{\sigma \in [0,\tau]} \int_0^\tau |C(s+t,\sigma-t)| \, dt \leq M_{[0,\tau]} \tau < +\infty$$

and the interval...”.

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בְּיְהוָה חֲסִיתִי אֶל-אֲבוֹשָׁה לְעוֹלָם:

In te, Domine, speravi, non confundar in aeternum.

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