

# Stability analysis of delay models by pseudospectral methods



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# delay equations

*"A delay equation is a rule  
for extending a function of time towards the future  
on the basis of the (assumed to be) known past."*

# ODE vs DDE

- autonomous first order Ordinary Differential Equation

$$y'(t) = f(y(t)), \quad f: \mathbb{R}^d \rightarrow \mathbb{R}^d$$

- Delay Differential Equation with one discrete delay  $\tau > 0$

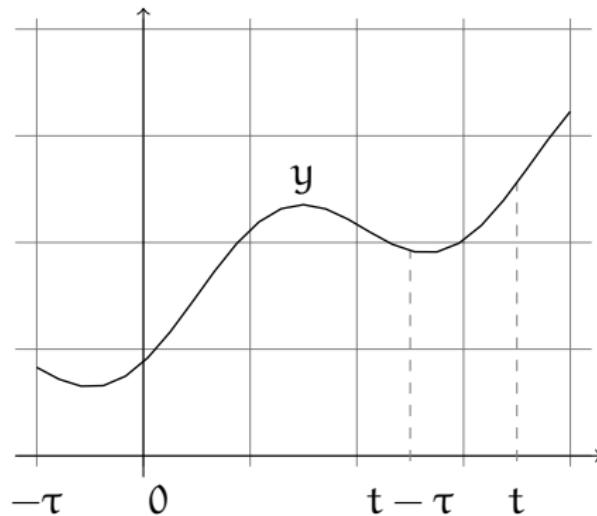
$$y'(t) = f(y(t), y(t - \tau)), \quad f: \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d$$

# history and infinite dimension

$$\tau > 0, \quad y: [-\tau, \infty) \rightarrow \mathbb{R}^d$$

$$y_t: [-\tau, 0] \rightarrow \mathbb{R}^d, \quad t \geq 0$$

$$y_t(\theta) := y(t + \theta)$$

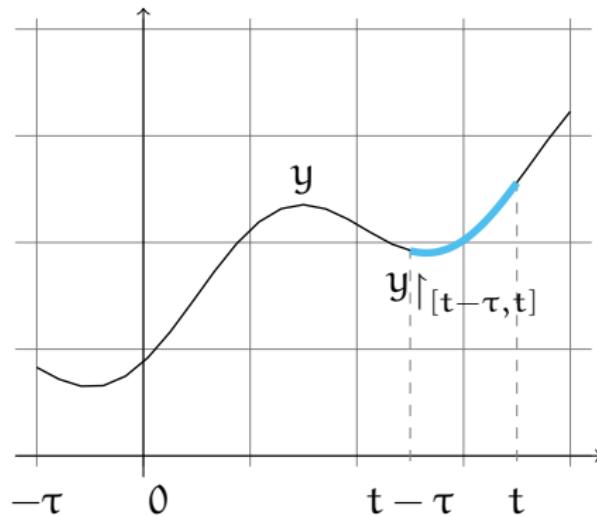


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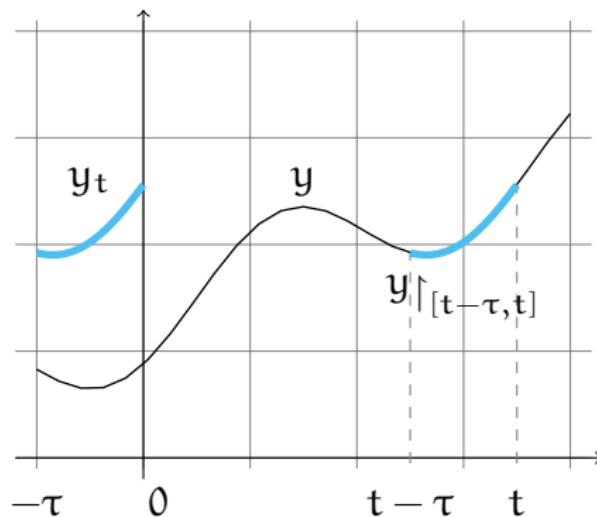


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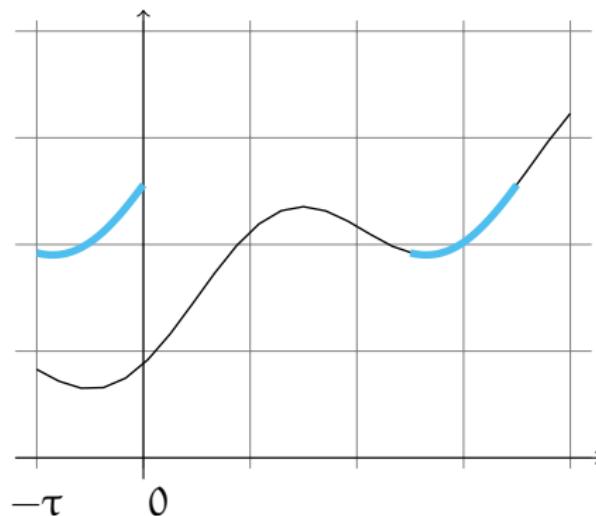


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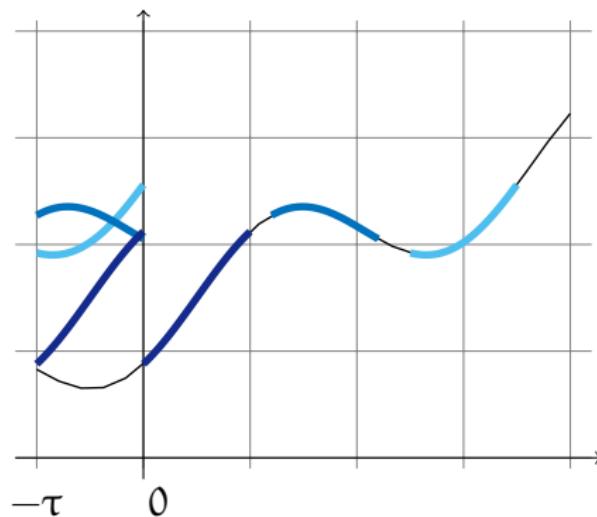


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# types of delay equations

$$\tau > 0, \quad x_t \in X = L^1([-\tau, 0], \mathbb{R}^{d_x}), \quad y_t \in Y = C([-\tau, 0], \mathbb{R}^{d_y})$$

- Delay Differential Equations (DDE)

$$y'(t) = G(y_t), \quad G: Y \rightarrow \mathbb{R}^{d_y}$$

- Renewal Equations (RE)

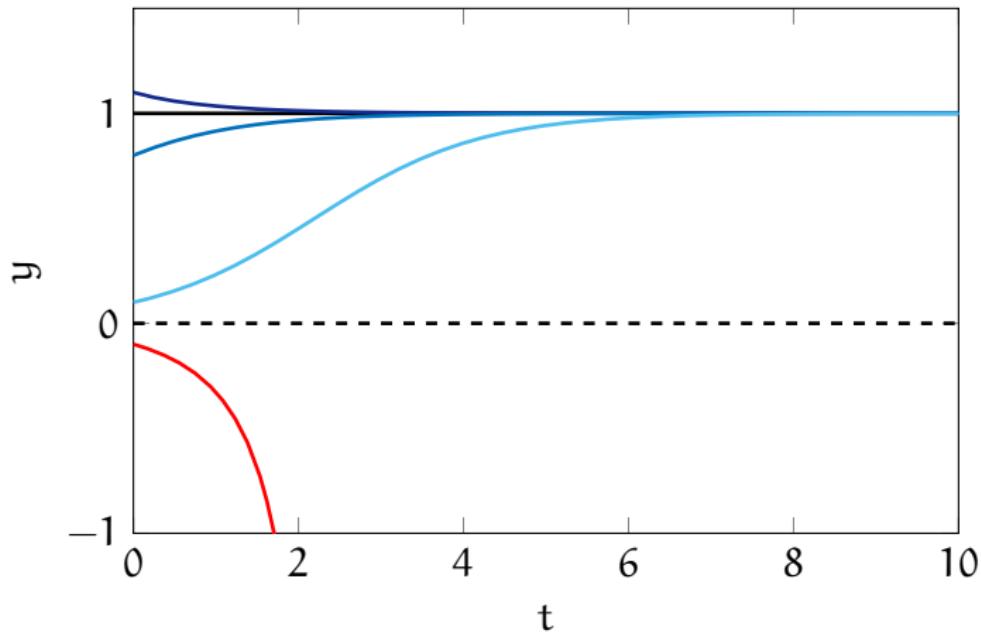
$$x(t) = F(x_t), \quad F: X \rightarrow \mathbb{R}^{d_x}$$

- coupled RE/DDE

$$\begin{cases} x(t) = F(x_t, y_t), & F: X \times Y \rightarrow \mathbb{R}^{d_x} \\ y'(t) = G(x_t, y_t), & G: X \times Y \rightarrow \mathbb{R}^{d_y} \end{cases}$$

# stability

$$y'(t) = y(t)(1 - y(t))$$



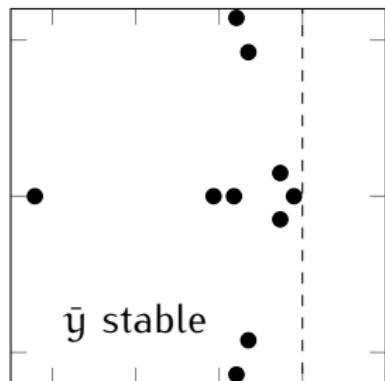
# stability for ODE: equilibria

$y'(t) = f(y(t)), \quad f: \mathbb{R}^d \rightarrow \mathbb{R}^d, \quad \bar{y}$  equilibrium

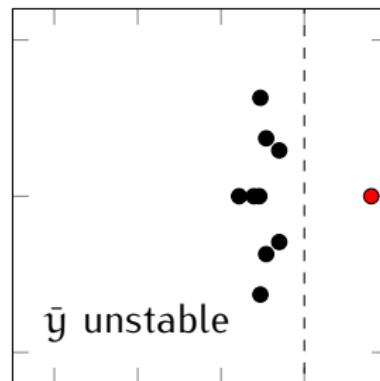
linearize around  $\bar{y} \rightsquigarrow y'(t) = Ay(t)$

$$A = Df(\bar{y}) = \begin{pmatrix} \frac{\partial f_1}{\partial y_1}(\bar{y}) & \cdots & \frac{\partial f_1}{\partial y_d}(\bar{y}) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_d}{\partial y_1}(\bar{y}) & \cdots & \frac{\partial f_d}{\partial y_d}(\bar{y}) \end{pmatrix} \in \mathbb{R}^{d \times d}$$

characteristic roots (CRs): eigenvalues of  $A$



$\bar{y}$  stable



$\bar{y}$  unstable

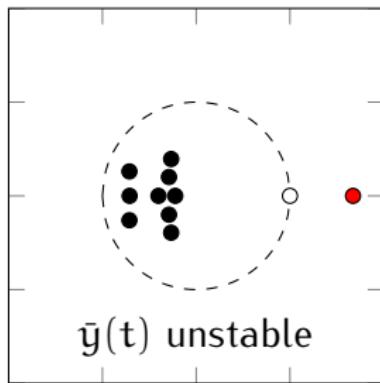
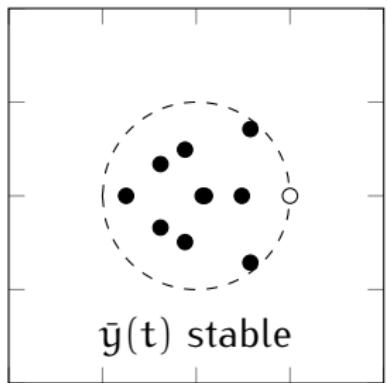
# stability for ODE: periodic solutions

$y'(t) = f(y(t)), \quad f: \mathbb{R}^d \rightarrow \mathbb{R}^d, \quad \bar{y}(t) \text{ } \Omega\text{-periodic solution}$

linearize around  $\bar{y}(t) \rightsquigarrow y'(t) = A(t)y(t), \quad A(t + \Omega) = A(t)$

Floquet theory  $\rightsquigarrow$  monodromy matrix  $M \in \mathbb{R}^{d \times d}$

characteristic multipliers (CMs): eigenvalues of  $M$



# stability for DDE: equilibria

linearize DDE around equilibrium  $\rightsquigarrow$  linear autonomous DDE

$$\begin{cases} y'(t) = Ly_t, & t \geq 0, \\ y_0 = \psi \end{cases} \quad L: Y \rightarrow \mathbb{R}^{d_Y}$$

$C_0$ -semigroup of solution operators  $T(t)y_0 = y_t, \quad t \geq 0$

infinitesimal generator of  $\{T(t)\}_{t \geq 0}$   $\mathcal{A}: \mathcal{D}(\mathcal{A}) \subseteq Y \rightarrow Y$

$$\mathcal{A}\psi = \psi'$$

$$\mathcal{D}(\mathcal{A}) := \{\psi \in Y \mid \psi' \in Y \text{ and } \psi'(0) = L\psi\}$$

abstract Cauchy problem  $\begin{cases} u'(t) = \mathcal{A}u(t), & t \geq 0, \\ u(0) = \psi \in Y \end{cases}$

$$u(t)(\theta) = y(t + \theta), \quad t \geq 0, \quad \theta \in [-\tau, 0]$$

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(ODE) Cauchy problem  $\begin{cases} y'(t) = Ay(t), & t \geq 0, \quad y(t) \in \mathbb{R}^d \\ y(0) = \hat{y} \in \mathbb{R}^d \end{cases}$

abstract Cauchy problem  $\begin{cases} u'(t) = \mathcal{A}u(t), & t \geq 0, \quad u(t) \in Y \\ u(0) = \psi \in Y \end{cases}$

$$u(t)(\theta) = y(t + \theta), \quad t \geq 0, \quad \theta \in [-\tau, 0]$$

# stability for DDE: periodic solutions

linearize DDE around periodic solution  $\rightsquigarrow$  linear periodic DDE

$$\begin{cases} y'(t) = L(t)y_t, & t \geq s, \quad L(t): Y \rightarrow \mathbb{R}^{d_Y}, \\ y_s = \psi & L(t + \Omega) = L(t) \end{cases}$$

evolution operators  $T(t, s)y_s = y_t, \quad t \geq s$

Floquet theory  $\rightsquigarrow$  monodromy operator  $\mathcal{M} := T(\Omega, 0)$

# stability for RE

equilibria: infinitesimal generator



periodic solutions: Floquet theory

work in progress

numerical method



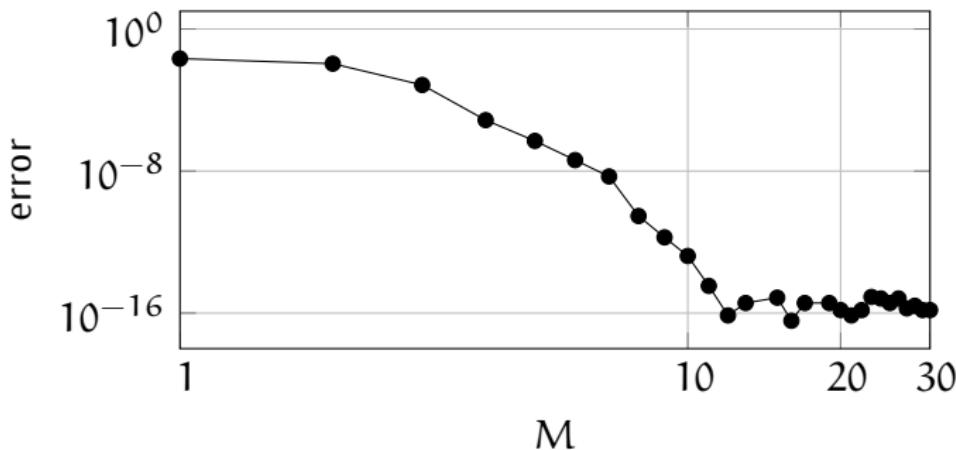
(paper in preparation)

# pseudospectral methods

$\psi \rightsquigarrow$  interpolating polynomial

spectral accuracy:  $\psi$  smooth  $\Rightarrow$  error =  $O(M^{-k})$  for every  $k$   
( $M$  = number of interpolation nodes)

example error plot



# a real-life problem: 2 host – 1 parasitoid model



*Drosophila melanogaster*, male



*Leptopilina heterotoma*



*Drosophila suzukii*, male

$$\left\{ \begin{array}{l} E'_i(t) = R_{E_i}(t) - M_{E_i}(t) - d_{E_i} E_i(t) \\ L'_i(t) = M_{E_i}(t) - M_{L_i}(t) - \alpha_i P(t) L_i(t) - d_{L_i}(L_i(t)) L_i(t) \\ A'_i(t) = M_{L_i}(t) - d_{A_i} A_i(t) \\ P'(t) = \sum_{i=1}^2 \alpha_i P(t - T_{iP}) L_i(t - T_{iP}) s_{iP} - d_P P(t) \end{array} \right.$$

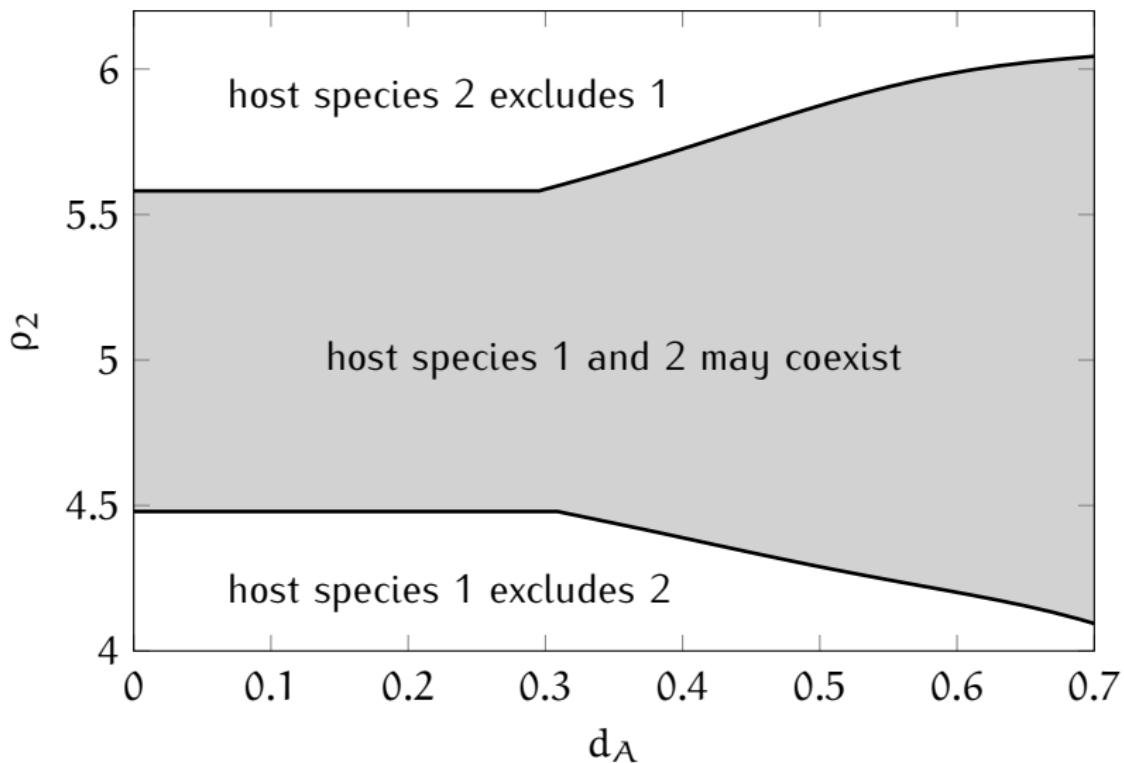
$$R_{E_i}(t) = \rho_i d_{A_i} A_i(t)$$

$$M_{E_i}(t) = R_{E_i}(t - T_{E_i}) e^{-d_{E_i} T_{E_i}}$$

$$M_{L_i}(t) = M_{E_i}(t - T_{L_i}) e^{-\int_{t-T_{L_i}}^t (\alpha_i P(\theta) + d_{L_i}(L_i(\theta))) d\theta}$$

# a real-life problem: 2 host – 1 parasitoid model

stability chart



## example 1: a simple linear DDE with one discrete delay

$$y'(t) = ay(t) + by(t - 1)$$

$$\left[ y'(t) = f(y(t), y(t - 1)) \right]$$

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$$\begin{aligned} \text{myDDE.m: } y'(t) &= \tilde{A}(t)y(t) + \sum_{u=1}^q \tilde{B}_u(t)y(t - d_u) \\ &\quad + \sum_{v=1}^w \int_{-l_v}^{-r_v} \tilde{C}_v(t, \theta)y(t + \theta) d\theta \end{aligned}$$

## example 1: a simple linear DDE with one discrete delay

some known facts from studying the characteristic equation

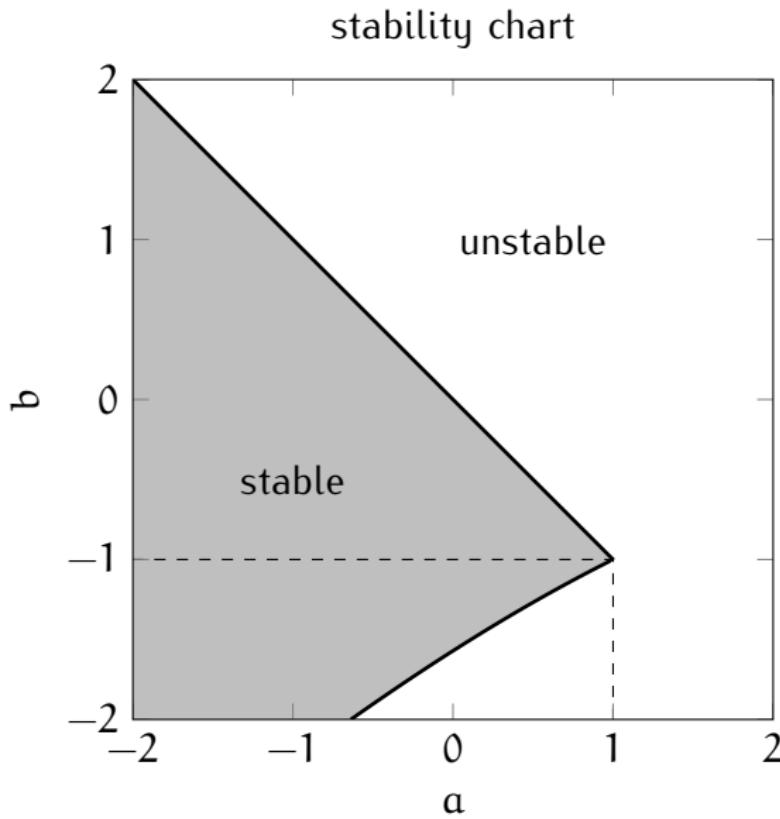
$$\lambda - a - b e^{-\lambda} = 0$$

$a = 2$  and  $b = -e \Rightarrow$  rightmost double CR 1

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$a$	rightmost CRs with $b = -\pi/2$
-1	complex conjugate pair, negative real part
0	imaginary pair $\pm i\pi/2$
1	complex conjugate pair, positive real part
$1 + \log(\pi/2)$	double real CR $\log(\pi/2)$
$3/2$	two real CRs, both positive
$\pi/2$	two real CRs, one 0 and one positive
2	two real CRs, one negative and one positive

# example 1: a simple linear DDE with one discrete delay



## example 2: a special RE with a quadratic nonlinearity

$$x(t) = \frac{\gamma}{2} \int_1^3 x(t-\theta)(1-x(t-\theta)) d\theta, \quad \gamma \geq 0$$

equilibria:

$$\bar{x}_1 = 0, \quad \bar{x}_2 = 1 - \frac{1}{\gamma}$$

periodic solutions for  $\gamma \geq 2 + \frac{\pi}{2}$ , all with period 4:

$$\bar{x}_3(t) = \frac{1}{2} + \frac{\pi}{4\gamma} + \alpha \sin\left(\frac{\pi}{2}t\right)$$

$$\alpha^2 = \frac{1}{2} - \frac{1}{\gamma} - \frac{\pi}{2\gamma^2} \left(1 + \frac{\pi}{4}\right)$$

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linearization around  $\bar{x}(t)$ :

$$x(t) = \frac{\gamma}{2} \int_1^3 x(t-\theta)(1 - 2\bar{x}(t-\theta)) d\theta$$

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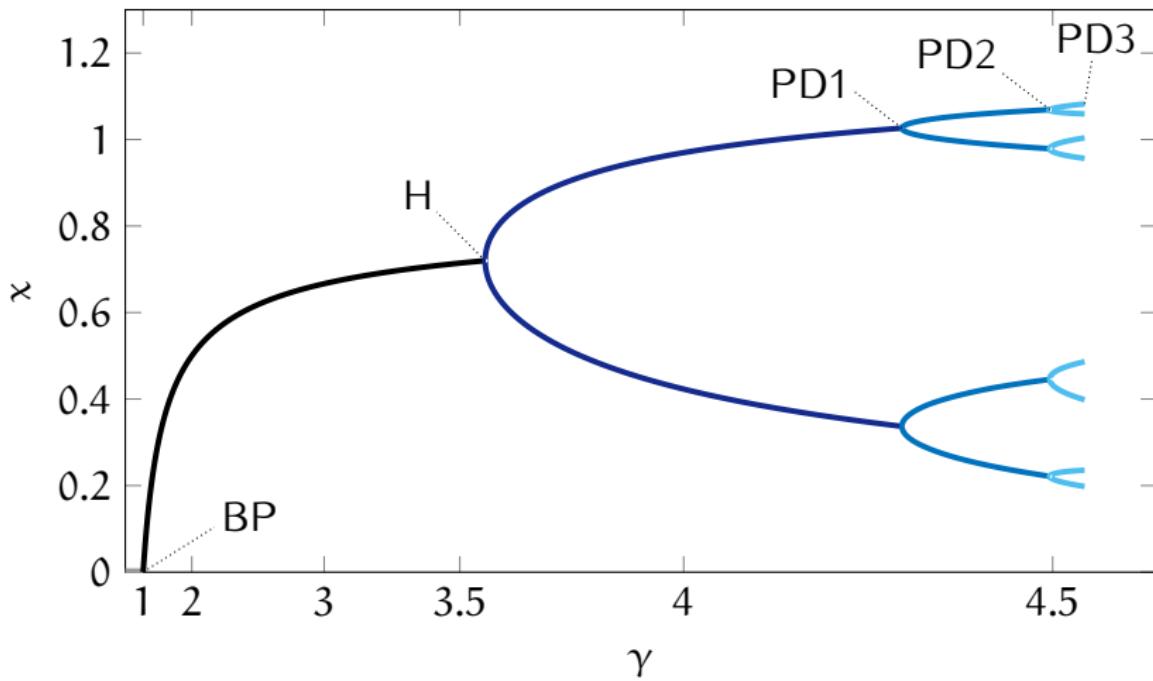
$$x(t) = \frac{\gamma}{2} \int_{-3}^{-1} x(t+\theta)(1 - 2\bar{x}(t+\theta)) d\theta$$

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$$\text{modelr.m: } x(t) = A(t)x(t) + \sum_{k=1}^p B_k(t)x(t-\tau_k) \\ + \sum_{k=1}^p \int_{-\tau_k}^{-\tau_{k-1}} C_k(t, \theta)x(t+\theta) d\theta \quad [\tau_0 = 0]$$

## example 2: a special RE with a quadratic nonlinearity

bifurcation diagram



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Photograph of *Leptopilina heterotoma*: HANS SMID, Laboratory of Entomology, Wageningen University, The Netherlands, <http://www.wur.nl/en/newsarticle/Parasitoid-wasps-can-count-hidden-competitors-through-taste-sensors-.htm>.

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